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Technical Report No. 32-797

Particle Symmetries

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CONTENTS

FIGURES

15411

ABSTRACT

A theory of particle symmetries is proposed based on general principles of quantum mechanics and special relativity. Starting with a modest generalization of the Poincaré group and using techniques of group theory and operator algebras, it is shown how to construct composite-particle state vectors labeled by external (i.e., pertaining to space-time properties) and internal quantum numbers of physical significance. Macroscopic space-time behaves in exactly the same manner under both the Poincaré group and its generalization, the augmented Poincaré group. It is found that there exists a hierarchy of groups, $Sp\,(1) \subset Sp\,(2) \subset Sp\,(3) \subset \cdots$, which characterizes internal symmetries of the composite particles. These groups are all noncompact. However, it is argued that *physical* particle states are characterized by the compact subhierarchy of unitary groups $U\,(1) \subset U\,(2) \subset U\,(3) \subset \cdots$. Thus, it is shown that the essential features of fundamental-particle symmetries can be derived in a general way from basic properties of space-time. These results are believed to form a theoretical framework for attacking dynamical problems such as the correlation of masses and spins with internal quantum numbers furnished by the hierarchy of unitary groups.

author

I. INTRODUCTION

This report is the first of a projected series in which we attempt to construct a theory of fundamental particles[1] and their interactions starting with a minimum of axioms. We assume that fundamental physical processes are governed by the laws of quantum mechanics and special relativity. This much, and usually more, is of course postulated in most relativistic particle theories. The distinguishing feature of our work is its mathematical methodology, which may be summarized by the phrase "representation theory of groups on Hilbert spaces." The two concepts, group and Hilbert space, respectively embody the mathematical essence of special relativity and quantum mechanics. It is only natural that they should be the primary objects of attention.[2]

The art of theoretical physics has reached the stage where it is no longer necessary for physicists to apologize for the mathematical sophistication of techniques used in

[1]By "fundamental particles" (rather than "elementary," an adjective to be used later in a more technical sense) we mean photons, leptons, hadrons, nuclei, etc.

[2]Proposals to make a systematic study of representations of the Poincaré group of special relativity have been made by Dirac. See Ref. 1.

the solution of physical problems. Still, simplicity and elegance of physical and mathematical ideas are to be strived for even though not always possible to attain. Keeping these two desiderata in mind, we have decided to explore the possibilities of formulating a theory of particles based not on the traditional notion of quantum fields but directly on operator algebras associated with representations of the group of special relativity. One of the motives for attempting this task is our desire to introduce a new and hitherto untried approach to particle physics in the hope that practical calculational schemes will eventually emerge.[3]

There are several fundamental problems facing any comprehensive theory of particles. Briefly stated, they are as follows. First, there is the problem of the origin of so-called internal symmetries associated with quantum numbers characterizing particles observed in nature.[4] Secondly, one is faced with the formulation and solution of the stability problem. More specifically, one would like to understand why only a small subset of all possible states apparently allowed by quantum mechanics and special relativity are realized in nature in the form of reasonably long-lived particles. The two problems taken together might be said to constitute that of calculating the "mass spectrum" of particles. Thirdly, the known hierarchy of strong, electromagnetic, weak, and gravitational interactions should find a theoretical explanation. Associated with this is the fourth problem, discovering the reason for the known striking correlations between the strengths of different interactions and the various experimentally observed conservation laws. The last major problem is of course the calculation of scattering amplitudes for strong-interaction physics, where perturbational techniques apparently fail and techniques based on dispersion theory are unbelievably complicated except in the simplest physical cases. All these problems are mutually interdependent, and it is difficult to see *a priori* how one could be solved without at least partially solving the others.

We shall not review the present theoretical situation concerning these problems except to note that some progress has been made in the solution of all of them save the third and the fourth; so far they remain unassailable. It is not that there is a lack of phenomenological theories correlating known experimental data. What we do not understand is the origin of the huge differences in the numerical values of coupling constants characterizing the different interactions. We shall offer some speculations on this matter in the last section of this report. In the meantime, we wish to examine more closely the first problem on the list.

There is an overwhelming amount of evidence that internal and external or space-time symmetries are intimately related. Hence there have been numerous attempts[5] to extend the external symmetry group, the Poincaré group of special relativity, in such a way as to accommodate internal symmetries within its fold. This procedure meets with the great difficulty of reconciling the observed particle multiplet mass splittings with the invariance of internal quantum numbers under Poincaré (or inhomogeneous Lorentz) transformations (see Ref. 3–6). This difficulty has apparently been resolved by an extension scheme recently suggested by Ottoson *et al.* (Ref. 7). In any case, such proposed group extensions do not really explain internal symmetries in any fundamental way; they merely lump external and internal symmetry groups together into a "supergroup." A more satisfying explanation of internal symmetries would be obtained if one could show how they originate from the four-dimensional space-time, if that is indeed their origin. Thus attempts are currently being made[6] to derive internal symmetries of strongly interacting particles by means of self-consistent calculations in the spirit of the bootstrap philosophy (see Ref. 9). The results of these attempts are admittedly encouraging although far from conclusive. The difficulty is of course that bootstrap calculations are very strongly model-dependent because of the still primitive state of strong-interaction dynamical theory. The thought occurs that perhaps the solution of the internal symmetries problem should be looked for elsewhere. We know that external states of free particles are determined purely by kinematics. Thus all positive values of m^2, the eigenvalue of one of the Casimir operators of the Poincaré group (see Section III), are kinematically possible;[7] however, only a very small subset of positive-m^2 representations are experimentally seen as free stable particles. Could it be that a similar situation occurs in the realm of internal symmetries? That is to say, could it be that *possible*, although not necessarily *physically realized*, internal states are also determined by considerations independent of dynamics? Putting it yet another way, is it possible to

[3]We have in mind possible alternatives to the dispersion-theoretic approach to strong-interaction dynamics.

[4]External quantum numbers such as mass and spin are adequately explained as being invariants of the Poincaré group.

[5]For recent work on this matter see Ref. 2.

[6]A summary of these efforts is given by Ref. 8.

[7]We are restricting ourselves to unitary representations of the Poincaré group, since only these can correspond to stable physical systems (and only for mass squared $m^2 \geq 0$).

construct a physically realistic theory of internal particle symmetries, based solely on the geometrical properties of space-time, providing a basis upon which a dynamical theory of particles may later be built? We believe that the answer is affirmative; the remainder of this report is devoted to a substantiation of this belief.

As we have already mentioned, our theory is based on certain operator algebras originating from various representations of the Poincaré group or groups related to it. Broadly speaking, any operator theory has two aspects to it: algebraic and analytic. Here we shall be concerned with the first one. Future work of this series will treat the dynamics of particles; there the "analytic" properties of operators (such as boundedness, convergence, continuity, etc.) are of very great importance because one has to deal with matrix elements of operators. Our treatment of internal symmetries in this work might therefore be viewed by the more mathematically inclined readers as insufficiently rigorous since we fail to exhibit the domains and ranges associated with various (unbounded) operators. We intend to remedy this mathematical deficiency in future work. Our primary interest at present is the development, be it somewhat mathematically nonrigorous, of a *physical idea* to be formulated in the next section.

Briefly, the plan of the report is the following. In Section II we formulate our theory in an intuitive fashion emphasizing the basic physical and mathematical ideas. The notation and some definitions and results from the theories of Lie algebras and of unitary representations of the restricted Poincaré group form the topic of Section III. The basic group of our theory is derived and discussed in Section IV. Definitions and results of mostly auxiliary mathematical character on tensor product representations of groups are contained in Section V. The main results of this report are presented in Section VI; there we obtain a hierarchy of internal symmetry schemes and discuss the construction of state vectors characterized by external and internal quantum numbers. The final section is devoted to a discussion of our results as well as to speculations on their consequences.

II. PHYSICAL FORMULATION

The purpose of this section is to provide a physical and heuristic mathematical formulation of the theory. We shall strive to present an intuitive description of the physics involved, leaving the more precise mathematical development of the theory to the following sections.

The foundations of our work are the theories of quantum mechanics and special relativity, as already pointed out in the last section. Let us very briefly recall the basic concepts involved.[8] In the usual formulation of quantum mechanics, the state of a physical system is mathematically described by a ray $\boldsymbol{\phi}$ in some Hilbert space $\mathfrak{H}$, i.e., by the totality of vectors $e^{i\alpha}\phi$ in $\mathfrak{H}$, where α is real and ϕ is a unit vector (the norm $\|\phi\| = (\phi, \phi)^{1/2} = 1$). The inner product $(\boldsymbol{\psi}, \boldsymbol{\phi})$ gives the probability amplitude for finding the system in the state $\boldsymbol{\psi}$ given that it is in the state $\boldsymbol{\phi}$, the probability of this being $|(\boldsymbol{\psi}, \boldsymbol{\phi})|^2$. To each physical observable there corresponds an hermitian operator H acting on $\mathfrak{H}$; $(\boldsymbol{\phi}, H\boldsymbol{\phi}) = (\phi, H\phi)$ represents the expectation value of this observable in the state described by $\boldsymbol{\phi}$. The converse is not true: there exist hermitian operators corresponding to no physical observables (see Ref. 11 and 12).

The theory of special relativity is introduced in the following way. If S and S′ are two Lorentz frames, then we say that observers situated in these two frames are equivalent. It is now assumed that the physics is the same for all equivalent observers. Thus, if $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ represent the states of a physical system in the language of an observer in the frame S, and if $\boldsymbol{\phi}'$ and $\boldsymbol{\psi}'$ represent respectively those of an observer in S′, then it is assumed that

$$|(\boldsymbol{\psi}', \boldsymbol{\phi}')|^2 = |(\boldsymbol{\psi}, \boldsymbol{\phi})|^2$$

It follows (see Ref. 13) that the rays $\boldsymbol{\phi}$, $\boldsymbol{\psi}$ and $\boldsymbol{\phi}'$, $\boldsymbol{\psi}'$ are connected by either a unitary or an antiunitary transformation depending solely on the two frames S′ and S:

$$\begin{aligned} \boldsymbol{\phi}' &= T_L\boldsymbol{\phi} \\ \boldsymbol{\psi}' &= T_L\boldsymbol{\psi} \end{aligned} \tag{1}$$

[8] For a more extensive review see Ref. 10.

where L is the (inhomogeneous, in general) Lorentz transformation from S to S'. If S'' is yet another frame with rays ϕ'' and ψ'' describing our physical system, and if L' and $L'' = L'L$ are the Lorentz transformations from S' to S'' and from S to S'', then one has

$$\phi'' = T_{L'} \phi'$$
$$\psi'' = T_{L'} \psi'$$

Combining these equations with Eq. (1), we can at most conclude that

$$T_{L''} = T_{L'L} = e^{i\alpha(L',L)}\, T_{L'} T_L, \quad \alpha \text{ real}$$

i.e., the operators T_L form an up-to-a-factor unitary representation[9] of the restricted Poincaré[10] group $P_0\,(= P_+^\uparrow)$ on the Hilbert space $\mathfrak{H}$. Only proper orthochronous Lorentz transformations, i.e., Lorentz transformations continuously connected to the identity, are assumed to be meaningful for macroscopic observers. This restriction may seem naive and even physically untenable; we shall see that this is not the case and that space-time inversions shall receive their due attention. As Wigner (Ref. 14) has shown, the up-to-a-factor unitary representations of P_0 may be replaced by the unitary representations of its universal covering group (see Ref. 15), provided that $|(\psi, T_L \phi)|^2$ is assumed to be continuous in L at the identity. Simply speaking, the only factors of significance are ± 1 associated with the single- and double-valued representations of P_0, as they are sometimes loosely called. The totality of all unitary representations of the restricted Poincaré group is the subject of our study.

Among all unitary representations of P_0, the irreducible ones play a dominant role. They are in a sense basic: an arbitrary unitary representation can be expressed in terms of the irreducible ones.[11] The fact that the state vectors of an irreducible representation of P_0 are characterized by fixed values of mass and spin (at least for the so-called "physical" representations) suggests that this representation somehow describes an "elementary" physical system. Indeed, this is the viewpoint of Newton and Wigner (Ref. 17) who *define* elementary physical systems as those whose state vectors transform irreducibly under the Poincaré group. As examples of such systems, we may mention the stable particles (we are assuming that they have infinite lifetimes): electrons, protons, neutrinos, photons, nonradioactive atoms in their ground states, etc. A neutron, on the other hand, is not an elementary physical system since it is not stable and hence is represented by a state vector with a slightly complex rest mass (see Ref. 18); such vectors do not belong to any irreducible unitary representation of P_0. For all practical purposes, however, the neutron and many other particles may conveniently be treated as elementary physical systems. Further, it is clear that stability is not synonymous with elementarity. Thus, e.g., the system of two free neutrinos of different momenta is certainly a stable system although not an elementary one. Neither has elementarity anything to do with the absence of an internal structure of a physical system. A proton, e.g., is an elementary physical system as pointed out above; its structure is revealed by various scattering experiments. In fact, there are several manifestations of internal structure of fundamental particles. First, we have the existence of so-called "nongeometric" quantum numbers such as the electric charge, baryon number, hypercharge, etc. It is believed, although not demonstrated, that they somehow characterize the internal structure of a particle; this we shall assume as a working hypothesis, henceforth calling these quantum numbers internal. We shall attempt to show that internal quantum numbers are of a purely geometric origin. Secondly, as already mentioned above, scattering experiments indicate an intricate charge distribution of particles such as nucleons. Finally, none of the particles are immutable: the interact, transform into each other, form bound states, decay, etc. It would be quite hard to see intuitively how all this could happen with structureless particles. In fact, the hypothesis of Chew and Frautschi (Ref. 19 and 20) represents the rather extreme view that all hadrons are composite, being bound states or resonances of each other. We adhere to their viewpoint, keeping in mind the possibility that *all* particles might be composite. This, we believe, is not unreasonable in view of the fact that leptons are known to have charges, magnetic moments, etc., just as the hadrons do.

Granted, then, the compositeness of elementary physical systems, we must find a way to describe their internal structure in an invariant manner, i.e., in a manner independent of the particular Lorentz frame of an overall physical system. To see how this may be done, let us take a simple example. Consider two spinless particles of momenta p_1 and p_2. If the two particles interact neither with themselves nor with other particles, then their state vector is just the (tensor) product of the state vectors of the individual particles: $\psi = \phi(p_1) \otimes \phi(p_2)$ Now ψ does not describe an elementary physical system because although ψ has a fixed rest mass, namely $m = [(p_1 + p_2)^2]^{1/2}$,

[9] Antiunitary transformations by themselves fail to form a group since the product of two such transformations is a unitary one.

[10] We use the by now standard nomenclature: Lorentz = homogeneous Lorentz, Poincaré = inhomogeneous Lorentz.

[11] This is of course not true for unitary representations of an arbitrary group; see Ref. 16.

it has no definite angular momentum. However, by taking a linear combination of the ψ's with various p_1 and p_2 subject to the restriction $p_1 + p_2 =$ fixed, one can build up a vector with a given integral angular momentum. We thus see that an elementary physical system can be constructed from a superposition of nonelementary ones mathematically represented by tensor products. This construction is just the inverse of the familiar mathematical procedure known as the reduction of a tensor product of two irreducible representations of a group into irreducible components. The reduction process yields vectors which transform irreducibly under the group in question and which in addition are labeled by certain quantum numbers invariant under all transformations of this group. These extra quantum numbers are necessary to remove the degeneracy inherent in the reduction process. To indicate concretely how this happens, let us consider the three-dimensional rotation group. The vectors of the $(2j + 1)$-dimensional irreducible (unitary) representation of this group are $\psi(jm)$, where $j(j + 1)$ is the eigenvalue of $\mathbf{J}^2$, the square of the total angular momentum operator, and m that of its z-projection J_z. The tensor product $\psi(j_1m_1) \otimes \psi(j_2m_2)$ can be written as a linear combination of the vectors $\psi(jm; j_1j_2)$ belonging to the different-j irreducible representations of the rotation group. The labels j_1 and j_2 are the "internal" quantum numbers of this "two-particle system"; it is clear that they do not change under all rotations generated by the "external" operator $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$. The ideas just outlined work just as well in the case of the restricted Poincaré group, or, for that matter, of any "reasonable" continuous group. It is clear that by reducing higher order tensor products, one obtains more and more internal quantum numbers. This should not be distressing since, after all, a particle, e.g., is a dynamical system with an infinite number of degrees of freedom (because it is a bound state of an arbitrary number of other particles including possibly itself). One may expect that these "higher order" internal quantum numbers should manifest themselves in future experiments at energies higher than are at present available. After all, it is a common phenomenon in physics that low-energy states of physical systems have the simplest possible quantum numbers.

We see in principle that by the processes of superposition (formation of linear combinations of vectors in $\mathfrak{H}$) and composition (formation of tensor products) we are able to generate irreducible unitary representations of the restricted Poincaré group whose basis vectors are labeled by certain internal quantum numbers invariant under external transformations of this group. For a given such representation of P_0, say one labeled by m and s, there exist infinitely many distinct sets of basis vectors each labeled by different internal quantum numbers. Thus although these representations are all equivalent under P_0, i.e., they all have the same transformation properties under this group, they are by no means physically equivalent (see p. 167 of Ref. 1). Indeed, the internal quantum numbers serve to indicate the "internal" state or configuration of an elementary (under P_0) physical system. It is plausible that just as there is the group P_0 associated with the external quantum numbers (describing the "center of mass" or "bulk" properties of a physical system such as mass, momenta, spin, etc.), there might also be a group or some similar mathematical object associated with internal quantum numbers. As we shall see later, this is indeed true, although the derivation of this mathematical object is not quite trivial. The point is that the above procedure for generating internal quantum numbers, although straightforward to carry out for P_0, just does not yield anything resembling the apparent internal symmetries in nature; the matter is discussed in Appendix A. One is forced either to discard the whole idea of generating internal symmetries by the processes of composition and superposition or to look for a generalization of P_0. We have chosen the latter alternative.

Given two vectors ϕ and ψ in the Hilbert space $\mathfrak{H}$, their superposition $\alpha\phi + \beta\psi$ (α, β complex) is again in $\mathfrak{H}$ by the definition of the Hilbert space. Thus $\mathfrak{H}$ is closed under superposition. However, this is not true for the operation of composition because $\phi \otimes \psi$ is no longer in $\mathfrak{H}$. Since, according to our viewpoint, the operation of composition is basic, we must introduce a "super-Hilbert" space, $\mathfrak{H}^\infty$, in which this operation is closed. This is of course nothing new; such spaces are implicitly assumed in all many-body quantum theories. Mathematically, they are known as infinite tensor products of ordinary Hilbert spaces, and they have been studied by von Neumann (Ref. 21); more will be said about them in Section V.

Let us now turn to the question of which irreducible representations of P_0 may be expected to be significant. In the conventional theories of particles, one distinguishes between physical and unphysical representations. The physical representations are those with nonnegative mass squared; all others are unphysical and therefore are to be discarded. We cannot accept this viewpoint any more than we can accept the viewpoint that negative energy solutions of the Dirac equation are unphysical and therefore uninteresting and unacceptable. We shall build our theory on the premise that *all* irreducible unitary representations of P_0 are important if we are to understand the dynamics of particles; the selection of "physical" representations as the ones observed experimentally is to be understood on the basis of their stability. In support of our viewpoint, we may remark that the so-called imaginary mass representations of P_0 have been shown by Wick (Ref. 22) to be closely related to the Regge formalism (Ref. 23). It should also be fairly obvious that the attractive and repulsive forces between particles mediated by the exchange of virtual particles ($m^2 < 0$) may mathematically be interpreted as being associated with such representations. These observations lead us to believe, to emphasize the point, that a consistent incorporation of all irreducible unitary representations of the restricted Poincaré group should result in a theory which is just as physical as the correctly interpreted theory of negative energy solutions of the Dirac equation.

For the reasons indicated in the next to last paragraph and in order to facilitate the mathematical treatment of the various irreducible unitary representations of the restricted Poincaré group, we shall introduce an enveloping group for it. The idea is simple. The restricted Poincaré group P_0 is embedded into a larger group, called the augmented Poincaré group P,[12] of which P_0 is a proper subgroup. Each irreducible unitary representation of P provides a unitary representation of P_0 which is in general reducible, although it may be made irreducible by a proper choice of basis vectors. Under the transformations of P, the different irreducible unitary representations of P_0 are mixed in a "smooth" way. Thus, for example, within the framework of P, vectors corresponding to states of different mass may be transformed into each other in a continuous fashion. This amounts to an "analytic continuation" of state vectors in their momentum eigenvalues. If we assume that the S-operator of our theory commutes with all the transformations of P (it already does so with those of P_0), then by means of the transformations of P, one may establish a connection between S-matrix elements characterized by different values of Lorentz invariants constructed from particle momenta. In other words, we have a group-theoretical prescription for analytically continuing the S-matrix elements in their Lorentz-invariant arguments; how this prescription works in detail will be shown elsewhere. Whether or not our method of analytic continuation is physically meaningful can of course be decided only by comparing the results of our computations with experiment. It suffices to emphasize now that since the analyticity properties of scattering amplitudes reflect the dynamics of particles, in going over from the group P_0 to P we are in some sense "building in" the dynamics into our theory.

Two obvious questions arise: How do we determine the enveloping group P, and is it unique? We attempt to answer these questions in Section IV. It turns out that there exists a very *natural* way which can be used to extend the restricted Poincaré group P_0; this extension is both maximal (i.e., one cannot extend P_0 any further) and unique. The basic idea of the method is as follows. A unitary representation of P_0 on a Hilbert space $\mathfrak{H}$ associates a unitary operator T_L with each (inhomogeneous) Lorentz transformation L. Now T_L has the form

$$\exp(-ia^\mu P_\mu)\exp(-i\omega^{\nu\rho}M_{\nu\rho}/2)$$

where P_μ and $M_{\nu\rho}$ are, respectively, the generators of space-time translations and rotations; the numbers a^μ and $\omega^{\nu\rho}$ specify the amount of translation and rotation. The P_μ and $M_{\mu\nu}$ satisfy certain commutation relations but are otherwise not unique. If $\mathfrak{B} = \{P_\mu, M_{\mu\nu}\}$ and $\mathfrak{B}' = \{P'_\mu, M'_{\mu\nu}\}$ are two distinct sets of generators (i.e., $P_\mu \neq P'_\mu$, $M_{\mu\nu} \neq M'_{\mu\nu}$) satisfying identical commutation relations, then it is clear that they are both equally suitable in constructing unitary operators representing P_0. If there is no relation between $\mathfrak{B}$ and $\mathfrak{B}'$, then there is nothing more to say. If, on the other hand, the operators in $\mathfrak{B}$ and $\mathfrak{B}'$ are in some way related to each other, then we may expect that the study of such relations might have some mathematical and possibly physical significance. One may argue that since $\mathfrak{B}$ and $\mathfrak{B}'$ each lead to a complete class of unitary representations of P_0, given a unitary representation R constructed with the help of the generators from $\mathfrak{B}$ and another unitary representation R' constructed with the help of those from $\mathfrak{B}'$, there might exist a unitary transformation U connecting the two representations R and R'. In other words, given T_L in R and T'_L in R' (same L!), we might have

$$U^{-1}T_LU = T'_L, \qquad \text{all } L \text{ in } P_0 \tag{2}$$

This equation mathematically expresses the equivalence of the two unitary representations R and R' of P_0. Suppose R and R' are irreducible under P_0 and different. Then Eq. (2) says that they are equivalent. But this cannot be so unless R and R' are two equivalent unitary representations of some *larger* group P of which P_0 is a subgroup. Thus the existence of U such that Eq. (2) is satisfied for $R \neq R'$ under P_0 presupposes the existence of a group P such that R and R' are equivalent under it.

If we take T_L to be first a pure infinitesimal translation and then a pure infinitesimal rotation, then, to first order, we have

$$\begin{aligned} U^{-1}P_\mu U &= P'_\mu \\ U^{-1}M_{\mu\nu}U &= M'_{\mu\nu} \end{aligned} \tag{3}$$

In other words, we are led to study the set of all U taking P_μ into P'_μ and $M_{\mu\nu}$ into $M'_{\mu\nu}$ and preserving the

[12]This is *not* the extended Poincaré group P_1 (in our notation) which includes space-time reflections.

commutation relations. It should be clear that the U's form a group; this group is essentially the augmented Poincaré group P. More precisely, the U's are the unitary operators *representing* P (with certain qualifications to be noted later). The burden of Section IV will be to determine the most general form of these unitary operators and hence the group P itself.

As already mentioned, the introduction of P allows one to effect a "unification" of all irreducible unitary representations of the restricted Poincaré group P_0. It turns out that a proper subgroup of P plays a fundamental role in our theory. This basic group, denoted by T, has a very simple structure, and yet its representation theory is sufficiently rich to allow one to construct all irreducible unitary representations of P_0 and P by means of the processes of superposition and composition. The representation Hilbert space $\mathfrak{H}^\infty$ (see Section V for its definition) of T is the arena in which the development of our theory henceforth takes place. It will be seen that by superposition and composition we shall be able to construct state vectors characterized by both external and internal quantum numbers of physical significance.

III. THE POINCARÉ GROUP

In this section we shall briefly review the theory of the (restricted) Poincaré group[13] and of some of its irreducible unitary representations in order to establish the notation and to collect results which will be needed in the sequel.

The Lorentz space[14] L consists of all real four-vectors[15] $x = (x_0, x_1, x_2, x_3)$, $x_0 = t, c = 1$, together with the quadratic form Q defined for each pair $x, y \in L$:

$$Q(x, y) = g^{\mu\nu}(x_\mu - y_\mu)(x_\nu - y_\nu), \qquad \mu, \nu = 0, 1, 2, 3$$

The summation convention on repeated dummy indices is understood, and the components of the metric tensor are

$$g^{00} = -g^{ii} = 1, \qquad i = 1, 2, 3$$
$$g^{\mu\nu} = 0, \qquad \mu \neq \nu$$

The Greek and Latin indices shall run over 0, 1, 2, 3 and 1, 2, 3, respectively. The raising and lowering of indices is accomplished by means of $g^{\mu\nu}$ and $g_{\mu\nu}$, both being equal numerically for the same set of indices $\mu\nu$. We call x_0 and (x_i) the time and space parts of x and shall frequently write

$$(x_1, x_2, x_3) = \mathbf{x}$$

The scalar product $x \cdot y$ of $x, y \in L$ is defined by[16]

$$x \cdot y = g^{\mu\nu} x_\mu y_\nu = x_\mu y^\mu = x_0 y_0 - \mathbf{x} \cdot \mathbf{y}$$

The vector x is said to be timelike, lightlike (or null), or spacelike according as $x^2 = x \cdot x$ is greater than, equal to, or less than zero.

The Poincaré group P_1 (the usual notation is P) is the group of transformations of L into itself of the form

$$x \to x' = lx + a$$

or, in components,

$$x_\mu \to x'_\mu = l_\mu{}^\nu x_\nu + a_\mu$$

which leave Q invariant:

$$Q(x', y') = Q(x, y) \tag{4}$$

We denote the elements of P_1 by (a, l), where $a = (a_\mu)$ is a four-vector and l can be thought as a 4×4 matrix with components $l_\mu{}^\nu$, μ labeling the rows and ν the columns. The group law of P_1 is

$$(a', l')(a, l) = (l'a + a', l'l) \tag{5}$$

[13] For a more detailed exposition of this theory see Ref. 14 and also Ref. 24–28.

[14] The nomenclature used is that suggested by Ref. 29. The distinction between Lorentz and Minkowski spaces is that in the latter one has the imaginary coordinate $x_4 = ict = ix_0$.

[15] We choose to introduce L in terms of covariant four-vectors x_μ.

[16] The scalar product is determined once Q is given:

$$x \cdot y = \tfrac{1}{2}[Q(0, x) + Q(0, y) - Q(x, y)]$$

The identity is $(0,1)$ (0 = zero vector, 1 = unit matrix), and the inverse is given by

$$(a,l)^{-1} = (-l^{-1}a, l^{-1})$$

We note the decomposition

$$(a,l) = (a,1)(0,l) \tag{6}$$

valid for every $(a,l) \in P_1$. The set of all $(a,1)$ is an abelian normal subgroup of P_1, denoted by T_0, of space-time translations of L. The set of all $(0,l)$ forms the homogeneous Lorentz (or, briefly, Lorentz) group L_1, a subgroup of P_1 containing all space-time rotations of L. The group law (Eq. 5) shows that P_1 is a semi-direct product of T_0 and L_1:

$$P_1 = T_0 \dot{\times} L_1$$

The condition of Eq. (4) leads to the restrictions

$$\det l = \pm 1$$

$$|l_0{}^0| \geq 1$$

Of special interest to us is the normal subgroup P_0 (usually denoted by $P_+^\uparrow$) of P_1 which is the semi-direct product of L_0 $(= L_+^\uparrow)$ and T_0, the group L_0 consisting of all l satisfying

$$l_0{}^0 \geq +1$$

$$\det l = +1$$

The group P_0 is a connected Lie group (see Ref. 15), called the restricted Poincaré group.

As is well known, the study of representations of a Lie group can be reduced to that of the Lie algebra of its identity component (see Ref. 15) and to the group of discrete automorphisms of this algebra. An abstract Lie algebra $\mathfrak{L}$ is a nonassociative algebra (see Ref. 30) over a given field K with an operation [,] (the commutator bracket) defined for each pair of elements in $\mathfrak{L}$ and obeying the rules

(i) $A, B \in \mathfrak{L} \Rightarrow [A,B] \in \mathfrak{L}$ (closure)

(ii) $A \in \mathfrak{L} \Rightarrow [A,A] = 0$ (antisymmetry)

(iii) $A, B, C \in \mathfrak{L} \Rightarrow [(A,B),C] + [(B,C),A] + [(C,A),B] = 0$ (Jacobi identity)

A subset $\mathfrak{B}$ of $\mathfrak{L}$ is called a basis of $\mathfrak{L}$ if every element of $\mathfrak{L}$ can be expressed as a linear combination of elements of $\mathfrak{B}$ with coefficients in K. When dealing with the Poincaré group or its extensions in the following, the field K will be the real field except when stated explicitly to the contrary.

The concept of the (universal) enveloping algebra of a Lie algebra $\mathfrak{L}$ will play a fundamental role in our considerations. To define it, we first introduce the tensor algebra $\mathfrak{T}$ of $\mathfrak{L}$. It is an associative algebra over K whose basis consists of elements of the form

$$X_{i_1}X_{i_2}\cdots X_{i_n}, \quad X_{i_k} \in \mathfrak{B}, \quad 1 \leq k \leq n = 1,2,3,\cdots \tag{7}$$

If $Y \in \mathfrak{T}$, then one defines

$$[Y, X_{i_1}X_{i_2}\cdots X_{i_n}] = [Y, X_{i_1}]X_{i_2}\cdots X_{i_n} + X_{i_1}[Y, X_{i_2}]\cdots X_{i_n} + \cdots + X_{i_1}X_{i_2}\cdots [Y, X_{i_n}]$$

The operation $[Y,\cdot]$ is thus distributive and is analogous to differentiation or derivation. Let us denote by $\mathfrak{Y}$ the set of all $I \in \mathfrak{T}$ which are of the form

$$I = [A,B] - AB + BA, \qquad A, B \in \mathfrak{T}$$

Then it is easy to check with the help of the Jacobi identity that $[T, I] \in \mathfrak{Y}$ for each $T \in \mathfrak{T}$ and each $I \in \mathfrak{Y}$. Thus $\mathfrak{Y}$ is an ideal of $\mathfrak{T}$. The factor algebra $\mathfrak{T}/\mathfrak{Y}$, consisting of all elements of $\mathfrak{T}$ in which elements $I \in \mathfrak{Y}$ are identified with the zero element of $\mathfrak{T}$, is called the (universal) enveloping algebra $\mathfrak{E}$.[17] The foregoing construction of $\mathfrak{E}$ is logically necessary since, strictly speaking, the commutator bracket $[A,B]$ is *not* defined to be $AB - BA$ because the symbols AB and BA have no meaning within the Lie algebra $\mathfrak{L}$ itself. This circumstance of course does not occur when A and B are operators on some vector space; then AB is just the usual operator product.

For the basis $\mathfrak{B}_0$ of the Lie algebra $\mathfrak{P}_0$ of the restricted Poincaré group P_0 one usually takes the ten generators P_μ and $M_{\mu\nu} = -M_{\nu\mu}$ of space-time translations and rotations,[18] respectively. The generators are assumed to be hermitian in order that the operators $\exp(-i\alpha X)$, α real,

[17]It can easily be shown that $\mathfrak{E}$ itself is an infinite-dimensional Lie algebra.

[18]We use the same notation for abstract operators as well as for their representatives on some (as yet unspecified) linear vector space.

$X \in \mathfrak{B}_0$, representing group elements of P_0 be unitary. We have the familiar commutation relations:

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [M_{\mu\nu}, P_\rho] &= iP_{[\mu}g_{\nu]\rho} \qquad (\hbar = 1) \\ [M_{\mu\nu}, M_{\rho\sigma}] &= iM_{[\mu[\sigma}g_{\rho]\nu]} \end{aligned} \tag{8}$$

where bracketed indices denote antisymmetrizations; e.g.,

$$M_{[\mu[\sigma}g_{\rho]\nu]} = M_{\mu[\sigma}g_{\rho]\nu} - M_{\nu[\sigma}g_{\rho]\mu}$$

The enveloping algebra $\mathfrak{E}_0$ of $\mathfrak{P}_0$ is constructed in the way outlined above except that we take $-i$ times the basis elements of $\mathfrak{P}_0$ in forming the products (Eq. 7) in order that the coefficients of the basis elements of $\mathfrak{E}_0$ be real. The significance of $\mathfrak{E}_0$ is, among other things, that it contains operators which generate unitary representations of P_0. We recall that a unitary representation R of P_0 on a Hilbert space $\mathfrak{H}$ is a group law preserving mapping of P_0 into the group of unitary operators on $\mathfrak{H}$. Thus

$$R: (a, l) \rightarrow U(a, l)$$

It is readily verified by using the commutation relations (Eq. 8) and the identities of Appendix B that the unitary operators

$$\begin{aligned} U(a, l) &= U(a)\, U(l) \\ &= \exp(-ia \cdot P) \exp(-i\omega : M/2) \end{aligned}$$

satisfy Eq. (5), where

$$\begin{aligned} \omega : M &= \omega^{\mu\nu} M_{\mu\nu} \\ l_{\mu\nu} &= (e^{\omega})_{\mu\nu} \\ &= g_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{2!} g^{\rho\sigma} \omega_{\mu\rho}\omega_{\sigma\nu} + \cdots \end{aligned} \tag{9}$$

The set of all $U(a, l)$, $(a, l) \in P_0$, thus forms the group $U(P_0)$ isomorphic to P_0 and contained in $\mathfrak{E}_0$ as a subset.

The operators $U(a, l)$ act on vectors belonging to the Hilbert space $\mathfrak{H}$. A subspace $\mathfrak{H}'$ of $\mathfrak{H}$ is said to be invariant under $U(P_0)$ if $U(a, l)\,\phi \in \mathfrak{H}'$ for every $\phi \in \mathfrak{H}'$ and for all $(a, l) \in P_0$. If $\mathfrak{H}'$ is an invariant subspace of $\mathfrak{H}$ containing no other invariant subspaces under $U(P_0)$ except $\{0\}$ and itself, then $U(P_0)$ is said to act irreducibly on $\mathfrak{H}'$, and the representation R is said to be irreducible on $\mathfrak{H}'$. The problem of determining irreducible unitary representations of P_0 is thus the same as that of finding invariant subspaces of $\mathfrak{H}$. All the irreducible unitary representations of P_0 have been determined by Wigner (see Ref. 14), and are summarized in Table 1. The numbers m^2 and $-m^2 s(s+1)$,

Table 1. Irreducible unitary representation of restricted Poincaré group. Here, $m, \mu > 0$, sgn $x = x/|x|$, $c = W^2$, $h = |W_0/P_0|$, and $\alpha = W^2/P^2 = -s(s+1)$.

Primes denote "two-valued representations." The little group of zero-momentum representations is the [(3 + 1)-dimensional] Lorentz group L_0; its representations are discussed in Section IV.

$P^2 = m^2 > 0$		sgn P_0	s		Little Group
$P_\pm^{ms}$		± 1	0, 1, 2, ···		0_3^+ (the three-dimensional rotation group)
$P^{ms'}$		± 1	1/2, 3/2, 5/2, ···		
$P^2 = 0$	c	sgn P_0	sgn W_0	h	Little Group
$0_\pm^h$	0	± 1	± 1	0, 1, 2, ···	E_2 (the two-dimensional euclidean group)
$0^{h'}$	0	± 1	± 1	1/2, 3/2, 5/2, ···	
0^c	> 0	± 1	—	—	
$0^{c'}$	> 0	± 1	—	—	
$P^2 = -\mu^2 < 0$		α	sgn W_0	s	Little Group
$Q^{\mu\alpha}$		> 0	—	—	$L_3^{(3)}$ (the three-dimensional Lorentz group)
$Q^{\mu\alpha'}$		$> 1/4$	—	—	
$Q^{\mu s}$		—	± 1	0, 1, 2, ···	
$Q^{\mu s'}$		—	± 1	−1/2, 1/2, 3/2, ···	

where m and s are physically interpreted as mass and spin, are the eigenvalues of so-called *Casimir operators* (see Ref. 31) (invariants of P_0 or, more correctly, of $\mathfrak{E}_0$) belonging to $\mathfrak{E}_0$:

$$P^2 = P_\mu P^\mu$$
$$W^2 = W_\mu W^\mu$$

Here

$$W_\mu = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma \tag{10}$$

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1, (\mu\nu\rho\sigma) = \text{even permutation of } (0123); \\ -1, (\mu\nu\rho\sigma) = \text{odd permutation of } (0123); \\ 0, \text{otherwise} \end{cases}$$

The Casimir operators P^2 and W^2 commute with every basis element of $\mathfrak{P}_0$, hence with every element of $\mathfrak{E}_0$ and, in particular, of $U(P_0)$. It follows by Schur's lemma (see Ref. 32) that they are constant multiples of the identity operator on $\mathfrak{H}$ in every irreducible unitary representation of P_0. Accordingly, such representations are labeled by the eigenvalues of P^2 and W^2. However, not all irreducible representations are determined by these two Casimir operators alone. Further group invariants exist and are given in Table I. Here we shall review the representation theory of classes P^{ms} and $P_+^{ms\prime}$ mainly to introduce certain concepts which are necessary for further development of the theory.[19]

The positive mass squared, positive energy representations P_+^{ms} and $P^{ms\prime}$ are characterized by the values of mass $m = +(m^2)^{1/2} > 0$ and spin $s = 0, \frac{1}{2}, 1, \cdots$. These two numbers specify the subspace $\mathfrak{H}(m, s)$ of the Hilbert space $\mathfrak{H}$ of all unitary representations of P_0. Within each $\mathfrak{H}(m, s)$, one may choose a set of basis vectors which diagonalize operators commuting among themselves and, of course, with P^2 and W^2. In order that the basis vectors be non-degenerate within the framework of P_0, we must find a maximal abelian subalgebra of $\mathfrak{E}_0$[20] and use the eigenvalues of its basis operators to label our vectors in $\mathfrak{H}(m, s)$. One such subalgebra[21] has as its basis the operators P_μ and W_0. We denote their eigenvalues by p_μ and $|\mathbf{p}|\,h$ ($h = -s, -s+1, \cdots, s$ is the helicity). Since $p_0^2 = \mathbf{p}^2 + m^2$, we may eliminate m and take the state vectors $|p\,s\,h\rangle$ as the basis of $\mathfrak{H}(m, s)$. The four-vector p_μ is of course subject to the restriction $p^2 = m^2 > 0$ in this case. We adopt the normalization

$$\langle p's'h' | psh\rangle = \delta(p' - p)\,\delta_{s's}\delta_{h'h}$$
$$\delta(p' - p) = \delta(p_0' - p_0)\,\delta(\mathbf{p}' - \mathbf{p})$$

In other words, the basis vectors $|p\,s\,h\rangle$ are singular elements of $\mathfrak{H}(m, s)$ in the terminology of Appendix C.

The action of unitary operators representing P_0 is given by

$$U(a, l)\,|psh\rangle = e^{-ia\cdot lp} \sum_{h'} D^s_{h'h}(R_w(l, p))\,|lpsh'\rangle \tag{11}$$

where R_w is the Wigner rotation operator (see Ref. 14) given in Appendix D. The transformation properties of the vectors $|p\,s\,h\rangle$ given by Eq. (11) make use of the spherical functions $D^s_{h'h}$ of the three-dimensional rotation group 0^+ which is the *little group* of timelike momenta, i.e., the group of all $\Lambda \in L_0$ which leave p fixed: $\Lambda p = p$. The D-functions are likewise discussed in Appendix D. Unitary representations of P_0 with spacelike, lightlike, and zero momenta involve different little groups which are given in Table I.

We shall not discuss the extended Poincaré group P_1 here since we shall see in the next section that P_1 may be obtained from P_0 by adjoining to P_0 certain discrete automorphisms ("space-time reflections") of its Lie algebra, $\mathfrak{P}_0$.

---

[19]We follow the work of Shirokov (Ref. 24–28).

[20]This corresponds to what Dirac calls "a complete set of commuting observables"; see Ref. 33. We prefer the standard mathematical terminology since, in many cases, it is not at all clear whether the operators in a maximal abelian subalgebra of a given algebra indeed represent physical observables. In this connection, see Ref. 34–35.

[21]It should be clear that the choice of a maximal abelian subalgebra is in most cases not unique, as in the present case, for instance.

IV. THE AUGMENTED POINCARÉ GROUP

This section is devoted to the study of irreducible unitary representations of the restricted Poincaré group P_0 from a unified standpoint. In studying the automorphisms of the Lie algebra $\mathfrak{P}_0$ of this group we are led in a natural manner to consider an extension of P_0, the augmented Poincaré group P. The Lie algebra $\mathfrak{P}$ of P has a rather simple structure and a readily available interpretation of its new generators. We consider in detail the representation theory of P and especially that of one of its subgroups, T.

In accordance with the ideas of Section II, we wish to investigate the automorphisms of $\mathfrak{P}_0$, i.e., transformations of elements in $\mathfrak{P}_0$ leaving the commutation relations (Eq. 8) invariant. Clearly, only linear transformations with real coefficients need be considered (with proper regard to the i's in the commutators) if $\mathfrak{P}_0$ is to be carried into itself. It should also be clear that it is sufficient to specify the transformations for the basis elements of $\mathfrak{P}_0$ alone. Thus if $P_\mu \to P'_\mu$ and $M_{\mu\nu} \to M'_{\mu\nu}$, then we require that P'_μ and M'_μ satisfy the same commutation relations as P_μ and $M_{\mu\nu}$.

The various automorphisms of $\mathfrak{P}_0$ may be broadly divided into two classes: continuous and discrete. Let us first investigate the latter class. We write

$$P_\mu = (P_0, \mathbf{P})$$
$$M_{\mu\nu} = (\mathbf{M}, \mathbf{N})$$

where

$$\mathbf{M} = (M_1, M_2, M_3) = (M_{23}, M_{31}, M_{12})$$
$$\mathbf{N} = (N_1, N_2, N_3) = (M_{01}, M_{02}, M_{03})$$

and introduce the transformations

$$\sigma: \quad \begin{aligned} P_\mu &\to {}^\sigma P_\mu = P^\mu = (P_0, -\mathbf{P}) \\ M_{\mu\nu} &\to {}^\sigma M_{\mu\nu} = M^{\mu\nu} = (\mathbf{M}, -\mathbf{N}) \end{aligned} \qquad (12)$$

and

$$\tau: \quad \begin{aligned} P_\mu &\to {}^\tau P_\mu = -P^\mu = (-P_0, \mathbf{P}) \\ M_{\mu\nu} &\to {}^\tau M_{\mu\nu} = M^{\mu\nu} = (\mathbf{M}, -\mathbf{N}) \end{aligned} \qquad (13)$$

It is easy to see that the commutation relations (Eq. 8) are unchanged under these transformations; thus σ and τ are (discrete) automorphisms of $\mathfrak{P}_0$. The set of automorphisms σ, τ, ρ, and ε, where

$$\rho = \sigma\tau: \quad \begin{aligned} P_\mu &\to {}^\rho P_\mu = -P_\mu = (-P_0, -\mathbf{P}) \\ M_{\mu\nu} &\to {}^\rho M_{\mu\nu} = M_{\mu\nu} = (\mathbf{M}, \mathbf{N}) \end{aligned} \qquad (14)$$

and ε is the identity, together with the relations $\varepsilon^2 = \sigma^2 = \tau^2 = \rho^2 = \varepsilon$ and $\sigma\tau = \tau\sigma = \rho$, forms an abelian group, the four-group V (see Ref. 36). Now it is known that the factor group P_1/P_0 is isomorphic to V and that every element $p_1 \in P_1$ may be written in the form $p_1 = vp_0$ with $v \in V$ and $p_0 \in P_0$. We see that by taking the restricted Poincaré group P_0 and adjoining to it the discrete group of automorphisms of its Lie algebra $\mathfrak{P}_0$, we have obtained the extended Poincaré group P_1, a group of acknowledged significance in particle physics. We should not be too surprised if, by adjoining further automorphisms of $\mathfrak{P}_0$ to P_1, we should obtain an even larger group of physical significance.

Let us next introduce an anti-automorphism. Define the mapping

$$\gamma: \quad \begin{aligned} P_\mu &\to {}^\gamma P_\mu = -P_\mu \\ M_{\mu\nu} &\to {}^\gamma M_{\mu\nu} = -M_{\mu\nu} \end{aligned} \qquad (15)$$

We see that $\gamma^2 = \varepsilon$ and that γ changes the right hand sides of all commutator brackets (Eq. 8) into their negatives or takes i into $-i$; this is precisely what we mean by an anti-automorphism. It should be clear that σ, τ, ρ, and γ correspond to the parity, strong (or Schwinger's) time reversal, strong reflection (or CPT), and charge conjugation (or, more correctly, particle-antiparticle conjugation) operations, respectively (see Ref. 10). The Wigner or weak time reversal (see Ref. 13) transformation T is just $\tau_w = \tau\gamma$. We shall have more to say about the discrete automorphisms and γ later on.

To begin the discussion of continuous automorphisms of $\mathfrak{P}_0$, consider the transformation

$$A \to A' = U^{-1}(a, l)\, A U(a, l) \qquad (16)$$

for $A \in \mathfrak{B}_0$, the basis of $\mathfrak{P}_0$, and $(a, l) \in P_0$. This is a mapping of an element of $\mathfrak{P}_0$ into an element of $\mathfrak{E}_0$. Using Eq. (8) and (B-2), we find

$$U^{-1}(a, l)\, P_\mu U(a, l) = l_\mu{}^\nu P_\nu$$
$$U^{-1}(a, l)\, M_{\mu\nu} U(a, l) = l_\mu{}^\rho l_\nu{}^\sigma (M_{\rho\sigma} + P_{[\rho} a_{\sigma]})$$

Thus A' is even in $\mathfrak{P}_0 \subset \mathfrak{E}_0$. It is easy to see that the mapping (Eq. 16) is an automorphism of $\mathfrak{P}_0$. Furthermore, every Casimir operator of P_0 is clearly unaffected by this mapping. In other words, irreducible unitary representations of P_0 are *not mixed* by these transformations; they merely effect some kinematical changes of state vectors and otherwise do not give anything new.

For $A \in \mathfrak{P}_0$, the mapping

$$\mathrm{ad}\,A: \quad B \to [A, B] \equiv \theta(A)\,B, \qquad B \in \mathfrak{P}_0$$

of $\mathfrak{P}_0$ into itself is called the adjoint mapping determined by A; here $\theta(A)\,B$ is called the Lie derivative of B with respect to A (see Appendix B). One can verify that Eq. (16) may be written as

$$A \to A' = E(a, l)\,A \tag{17}$$

where

$$E(a, l) = \exp[i\theta(a \cdot P)] \exp[i\theta(\omega{:}M/2)]$$

with $l = e^{\omega}$ as given by Eq. (9). The set of all $E(a, l)$ forms the group $\mathrm{Aut}_0(\mathfrak{P}_0)$, a subgroup of the group $\mathrm{Aut}(\mathfrak{P}_0)$ of all automorphisms of $\mathfrak{P}_0$. The group law of $\mathrm{Aut}_0(\mathfrak{P}_0)$ is just that of P_0, i.e., $\mathrm{Aut}_0(\mathfrak{P}_0) \cong$ (isomorphic to) P_0. The only elements of $\mathrm{Aut}(\mathfrak{P}_0)$ not in $\mathrm{Aut}_0(\mathfrak{P}_0)$ available to us so far are the discrete automorphisms in V. If $\omega \in V$, then, e.g.,

$$\omega^{-1}\,\theta(a \cdot P)\,\omega = \theta(a \cdot {}^{\omega}P) = \theta({}^{\omega}a \cdot P)$$

so that

$$\omega^{-1}\,E(a, l)\,\omega = E({}^{\omega}a, {}^{\omega}l) \in \mathrm{Aut}_0(\mathfrak{P}_0)$$

That is to say, $\mathrm{Aut}_0(\mathfrak{P}_0)$ is a normal or an invariant subgroup of $\mathrm{Aut}_0(\mathfrak{P}_0) \times V$. The $E(a, l)$ are accordingly called invariant automorphisms (Ref. 30) of the group $\mathrm{Aut}_0(\mathfrak{P}_0) \times V \subset \mathrm{Aut}(\mathfrak{P}_0)$. The notion of invariant automorphisms will be important in characterizing the augmented Poincaré group. Let us continue our search for further automorphisms of $\mathfrak{P}_0$.

Consider the scale transformations S_α defined by

$$S_\alpha: \quad \begin{aligned} P_\mu &\to \alpha P_\mu \\ M_{\mu\nu} &\to M_{\mu\nu} \end{aligned}$$

where α is a nonzero real number; it is clear that the commutation relations (Eq. 8) are unchanged under this transformation. The set of all such S_α is called the scale group S[22] of automorphisms of $\mathfrak{P}_0$. It is an abelian group with the rules

$$S_\alpha S_{\alpha'} = S_{\alpha\alpha'}$$

$$I = S_1$$

$$S_\alpha^{-1} = S_{\alpha^{-1}}$$

One could generalize this group by allowing arbitrary nonzero complex values of α. This, however, would destroy the hermiticity of the translation generators of $\mathfrak{P}_0$ and would lead to non-unitary representations of P_0 which we wish to avoid, at least for the time being. Moreover, it is sufficient to consider the case $\alpha > 0$ since transformations with negative α can be written as products of $S_{-\alpha}$ and the CPT operator ρ.

As discussed in Section II, we should like to be able to write the transformation $P_\mu \to \alpha P_\mu$ in the form $U_\alpha^{-1} P_\mu U_\alpha = \alpha P_\mu$ with some unitary operator depending on α. This is easily accomplished if we introduce a formally hermitian operator D satisfying the commutation relations

$$[P_\mu, D] = iP_\mu$$

$$[M_{\mu\nu}, D] = 0$$

Then

$$U_\alpha = \exp(-i \log \alpha D)$$

is the required unitary (for $\alpha > 0$!) operator. To show that D indeed exists, we may choose the representation (spin zero case)

$$P_\mu = p_\mu$$

$$M_{\mu\nu} = i p_{[\mu} \partial_{\nu]}$$

where $\partial_\nu = \partial/\partial p^\nu$. Then the operator $-ip_\mu \partial^\mu$ satisfies the commutation relations of D and hence may be taken as its representative. Introduction of spin does not change our conclusions because any spin operator must commute with the orbital part $ip_{[\mu}\partial_{\nu]}$ of $M_{\mu\nu}$. We shall encounter the dilation operator D in a disguised form in Section VI.

It is interesting to note that the scale transformations do not leave P^2 invariant and hence *mix* the different-mass irreducible unitary representations of P_0. The introduction of scale transformations allows us to effect a "unification"

[22]This group has recently been discussed by several authors: see Wess (Ref. 37) and Kastrup (Ref. 38).

of these representations. It is obvious that this unification is of a very limited nature since, first, the sign of P^2 is preserved ($\alpha > 0$ for unitary scale transformations, as noted previously), and, secondly, the spin eigenvalues are unaffected by D. We must therefore look for automorphisms of $\mathfrak{P}_0$ by means of which each component P_μ may be transformed independently of the others so that one may obtain a mixing of time-, light-, and spacelike vectors. We then may hope that the same automorphisms will allow us to unify the different-spin representations of P_0. We shall see that our hopes will be fulfilled.

In constructing a theory of particles based on unitary representations of P_0 we must not *a priori* eliminate the up-to-a-factor representations of P_0. This elimination is certainly justified by Wigner's theorem (see Ref. 14) whenever we study any *single given* representation but should not be expected to be meaningful when we consider the totality of all unitary representations of P_0 and the relations between them. In other words, we should be concerned about the relative phases of vectors belonging to various representations of P_0. A simple way of doing this is to enlarge the Lie algebra $\mathfrak{P}_0$ slightly by adding to it an identity operator, I, commuting with all the elements of $\mathfrak{P}_0$. Denote the resulting Lie algebra by $\mathfrak{P}_0'$. The Lie group P_0' corresponding to $\mathfrak{P}_0'$ is the direct product of P_0 and the group U_1 of complex numbers of unit modulus. As we shall see, the introduction of I has far-reaching consequences.

We now wish to study the automorphisms of $\mathfrak{P}_0'$. Consider first those of the subalgebra $\mathfrak{T}_0' = \mathfrak{T}_0 \oplus \{I\}$ of $\mathfrak{P}_0'$. The most general transformation of momentum four-vectors is given by

$$P_\mu \rightarrow P_\mu' = a_\mu^\nu P_\nu + v_\mu I$$

where a_μ^ν is a product of a scale factor $\alpha > 0$, a proper orthochronous Lorentz transformation $l_\mu{}^\nu$, and an element $\omega \epsilon V \times \{\gamma\}$, and v_μ is a real number. The homogeneous transformations $P_\mu \rightarrow a_\mu^\nu P_\nu$ have already been discussed; let us concentrate on the inhomogeneous case $P_\mu \rightarrow P_\mu + v_\mu I$. In order that this automorphism be of the form of Eq. (3) or (17), we introduce the operators X_μ by setting

$$E(v) P_\mu = e^{iv\cdot X} P_\mu e^{-iv\cdot X} = P_\mu + v_\mu I \qquad (18)$$

Expanding this expression in terms of v, we find

$$[P_\mu, X_\nu] = ig_{\mu\nu} I$$

The set $\{I, P_\mu, X_\mu, M_{\mu\nu}\}$ forms the basis for a new Lie algebra, $\mathfrak{P}$, provided $X = (X_\mu)$ is a vector operator and its components commute:

$$[M_{\mu\nu}, X_\rho] = iX_{[\mu} g_{\nu]\rho}$$
$$[X_\mu, X_\nu] = 0$$

This is necessary in order that the Jacobi identity be satisfied. In view of its commutation relations, especially those with P_μ, the vector X may be considered as a relativistic time-position (or four-position) operator.[23] We see by Eq. (18) that it generates momentum displacements. Conversely, the identities

$$E(a) X_\mu = e^{ia\cdot P} X_\mu e^{-ia\cdot P} = X_\mu - a_\mu I$$

show that P generates position displacements. Thus there exists a sort of duality between P and X. We shall have more to say about this later. The specification of the displacement automorphisms generated by $E(v)$ and given by Eq. (18) must be supplemented by indicating their action on $M_{\mu\nu}$:

$$E(v) M_{\mu\nu} = M_{\mu\nu} + X_{[\mu} v_{\nu]}$$

It is now clear that although $P_\mu \rightarrow P_\mu' + v_\mu I$ is an automorphism of $\mathfrak{T}_0'$, it cannot be one of $\mathfrak{P}_0'$ since by the above equation it takes $M_{\mu\nu}$ into an element outside of $\mathfrak{P}_0'$. Thus, in a way, we are *forced* to enlarge our original Lie algebra $\mathfrak{P}_0'$ in order to accommodate the automorphisms $E(v)$ and still obey the rules of the game by requiring that the X_μ be the basis elements of some Lie algebra.

We say that a Lie algebra $\mathfrak{L}$ is *complete* if each of its automorphisms continuously connected to the identity automorphisms is generated by some element of the enveloping algebra of $\mathfrak{L}$; i.e., all such automorphisms may be written as

$$\mathfrak{L} \rightarrow E_X \mathfrak{L}$$
$$E_X = \exp(i\theta(X))$$

for some X in the enveloping algebra. Each automorphism of $\mathfrak{L}$ is then generated by some ωE_X, where ω is a discrete automorphism of $\mathfrak{L}$. It should be clear that $\mathrm{Aut}_0(\mathfrak{L})$, the group of invariant automorphisms of $\mathfrak{L}$, is

[23]This appellation, though simple and concise, is somewhat misleading since the X_μ *are not* position operators of *physical particles*. The reason for this is that the X_μ do not leave invariant the physical subspace of positive energy state vectors. In connection with this subject see Ref. 17 and Ref. 39–41.

isomorphic to the group $\{E_X : X \text{ in the enveloping algebra of } \mathfrak{L}\}$. A complete Lie algebra $\mathfrak{L}$ has the desirable property that all its continuous automorphisms are invariant and are generated by combinations of operators already in $\mathfrak{L}$; in a sense, $\mathfrak{L}$ is self-sufficient and cannot be extended by the process exemplified in connection with $\mathfrak{P}_0'$. As we show in Appendix E, the *Lie algebra* $\mathfrak{P}$ *is complete.* Associated with $\mathfrak{P}$ is a connected Lie group which we call the augmented Poincaré group P. We believe that P is a group which is physically both relevant and important; we shall attempt to substantiate our belief in this and the following sections. We may remark that Segal (Ref. 42) has pointed out the possible physical significance of P, although his motivation for introducing and considering it is different from ours.

We now turn to the investigation of the structure of the augmented Poincaré group and to the determination of its irreducible unitary representations. We collect for convenience the commutation relations of the basis elements of $\mathfrak{P}$:

$$
\begin{aligned}
[I, P_\mu] &= [I, X_\mu] = [I, M_{\mu\nu}] = [P_\mu, P_\nu] \\
&= [X_\mu, X_\nu] = 0 \\
[P_\mu, X_\nu] &= i g_{\mu\nu} I \\
[M_{\mu\nu}, P_\rho] &= i P_{[\mu} g_{\nu]\rho} \\
[M_{\mu\nu}, X_\rho] &= i X_{[\mu} g_{\nu]\rho} \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i M_{[\mu[\sigma} g_{\rho]\nu]}
\end{aligned}
\tag{19}
$$

The set $\{X_\mu, M_{\mu\nu}\}$ is a basis of a Lie subalgebra $\mathfrak{P}_X$ of $\mathfrak{P}$ isomorphic to $\mathfrak{P}_P \equiv \mathfrak{P}_0$, the correspondence of course being $X_\mu \leftrightarrow P_\mu$ and $M_{\mu\nu} \leftrightarrow M_{\mu\nu}$. This isomorphism shows the previously mentioned duality between the momentum and position space representations of states.

A comment may be made regarding the identity operator of $\mathfrak{P}$. Since I commutes with every element of $\mathfrak{P}$, and since, by assumption, it is hermitian, it follows that its spectrum is the whole real line. We denote the eigenvalues of I by σ. It will be convenient to give σ an infinitesimal imaginary part: $\sigma = \sigma_0 + i\epsilon$, $-\infty < \sigma_0 < \infty$; the choice of the sign of ϵ is immaterial at the moment. Thus σ is never zero, and we may define the inverse I^{-1} (or $1/I$) of I as the operator whose eigenvalues are σ^{-1}.

The enveloping algebra $\mathfrak{E}$ of $\mathfrak{P}$ is constructed in the way already described in Section III. We shall assume that $\mathfrak{E}$ contains I^{-1} as well as other functions of I. Let us define

$$L_{\mu\nu} = X_{[\mu} P_{\nu]} / I \in \mathfrak{E} \tag{20}$$

One finds

$$
\begin{aligned}
[L_{\mu\nu}, P_\rho] &= i P_{[\mu} g_{\nu]\rho} \\
[L_{\mu\nu}, X_\rho] &= i X_{[\mu} g_{\nu]\rho} \\
[L_{\mu\nu}, M_{\rho\sigma}] &= [L_{\mu\nu}, L_{\rho\sigma}] = i L_{[\mu[\sigma} g_{\rho]\nu]}
\end{aligned}
\tag{21}
$$

The operators $L_{\mu\nu}$ have commutation relations similar to those of $M_{\mu\nu}$; in fact, the Lie algebra $\mathfrak{P}'$ generated by $\mathfrak{B}' = \{I, P_\mu, X_\mu, L_{\mu\nu}\}$ is isomorphic to $\mathfrak{P}$. From the last of Eq. (21) we see that the difference $M_{\mu\nu} - L_{\mu\nu}$ commutes with $L_{\rho\sigma}$. Thus it is natural to introduce new elements of $\mathfrak{E}$ by defining

$$S_{\mu\nu} = M_{\mu\nu} - L_{\mu\nu} \tag{22}$$

We immediately see that

$$
\begin{aligned}
[S_{\mu\nu}, I] = [S_{\mu\nu}, P_\rho] = [S_{\mu\nu}, X_\rho] = [S_{\mu\nu}, L_{\rho\sigma}] = 0 \\
[S_{\mu\nu}, S_{\rho\sigma}] = i S_{[\mu[\sigma} g_{\rho]\nu]}
\end{aligned}
\tag{23}
$$

The set $\mathfrak{B}'' = \{I, P_\mu, X_\mu, S_{\mu\nu}\}$ forms the basis for an especially simple Lie algebra, call it $\mathfrak{P}''$, in which the operators $S_{\mu\nu}$ are uncoupled from the remainder of the set. The $S_{\mu\nu}$ generate a Lie algebra isomorphic to that of the restricted Lorentz group L_0. The remaining nine operators I, P_μ, X_μ span a Lie algebra which we call the translation subalgebra $\mathfrak{T}$ of $\mathfrak{P}$. We see that $\mathfrak{P}''$ splits into a direct sum of two Lie algebras:

$$\mathfrak{P}'' \cong \mathfrak{T} \oplus \mathfrak{L}_0 \tag{24}$$

This is very pleasant: direct sums of Lie algebras correspond to direct products of Lie groups whose representations are just products of the representations of their individual factors. Of course, $\mathfrak{P}''$ is not isomorphic to $\mathfrak{P}_0$. However, their enveloping algebras are isomorphic, and this is all that matters since we are mainly interested in the unitary transformations contained in $\mathfrak{E}$. This means that we are free to use either of the basis sets $\mathfrak{B}$ or $\mathfrak{B}''$, whichever is more convenient in a particular circumstance. The connection between the two sets is provided by Eq. (20) and (22).

As it should be clear from their commutation relations, $L_{\mu\nu}$ and $S_{\mu\nu}$ are respectively the relativistic orbital and spin angular momentum tensor operators. Let us consider their relation to various other operators in $\mathfrak{E}$. Recall that the dual $\tilde{A}_{\mu\nu}$ of a tensor $A_{\mu\nu}$ is defined by the formula

$$\tilde{A}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} A^{\rho\sigma}$$

From this we find

$$A_{\mu\nu} = -\frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\,\tilde{A}^{\rho\sigma}$$

with the help of

$$\varepsilon_{\mu\nu\rho\sigma}\,\varepsilon^{\mu\nu\alpha\beta} = -2!\ \delta^{\alpha\beta}_{\rho\sigma} = -2\,(\delta_\rho^{\ \alpha}\delta^\beta_\sigma - \delta_\rho^{\ \beta}\delta^\alpha_\sigma)$$

From Eq. (10) and (22) we see that the polarization operator W_μ may be written as

$$W_\mu = \tilde{M}_{\mu\nu}P^\nu = \tilde{S}_{\mu\nu}P^\nu$$

since $\tilde{L}_{\mu\nu}P^\nu = 0$. It is possible to introduce another composition of $S_{\mu\nu}$ and P^ν:

$$Q_\mu = S_{\mu\nu}P^\nu$$

We find the following relations between the various operators:

$$S_{\mu\nu}P_\rho P^\rho = Q_{[\mu}P_{\nu]} - \varepsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma$$

$$\tilde{S}_{\mu\nu}P_\rho P^\rho = W_{[\mu}P_{\nu]} + \varepsilon_{\mu\nu\rho\sigma}Q^\rho P^\sigma$$

The vector operators W and Q lie in a hyperplane orthogonal to P:

$$W\cdot P = Q\cdot P = 0$$

they have a total of six independent components and hence may be used to replace the tensor operator $S_{\mu\nu}$ (if we allow division by $P_\rho P^\rho$). It should be mentioned that Q is related to the c.m. position operator $M_{\mu\nu}P^\nu/P_\rho P^\rho$ discussed by several authors (Ref. 43–45). We shall not make use of this fact.

Let us consider the structure of $\mathfrak{T}$ next. Writing

$$U(\alpha, v, a) = U(\alpha)\,U(v)\,U(a)$$

$$U(\alpha) = \exp(-i\alpha I)$$

$$U(v) = \exp(-iv\cdot X)$$

$$U(a) = \exp(-ia\cdot P)$$

and using the commutation relations (Eq. 19), we find

$$U(\alpha', v', a')\,U(\alpha, v, a) = U(\alpha' + \alpha + a'\cdot v,\ v' + v, a' + a) \tag{25}$$

Thus the unitary operators $U(\alpha, v, a)$ form a group. We *define* the translation subgroup T of P to be the group of all triples (α, v, a), where $-\infty < \alpha < \infty$ and v and a are real four-vectors, satisfying a group law which is the inverse image of Eq. (25).

$$(\alpha', v', a')(\alpha, v, a) = (\alpha' + \alpha + a'\cdot v, v' + v, a' + a) \tag{26}$$

In particular, the inverse elements of T are given by

$$(\alpha, v, a)^{-1} = (-\alpha + a\cdot v, -v, -a)$$

To have a more concrete characterization of T, we note that with each $(\alpha, v, a) \in T$ we may associate a real 6×6 matrix of the form

$$\begin{bmatrix} 1 & \tilde{v} & \alpha \\ 0 & I_4 & a \\ 0 & 0 & 1 \end{bmatrix} \tag{27}$$

where I_4 is the unit 4×4 matrix, a is a column vector with the entries a_0, a_1, a_2, a_3, top to bottom, and $\tilde{v}$ is the row vector $(v^0, v^1, v^2, v^3) = (v_0, -v_1, -v_2, -v_3)$. One can easily verify that the matrices (Eq. 27) satisfy the group law (Eq. 26). We call T the basic group of our theory for reasons which will become apparent in later sections. We note that $\mathfrak{T}$, the Lie algebra of T, is a relativistic generalization of the canonical commutation relations

$$[p_i, q_j] = -i\delta_{ij}$$

by the addition of a bracket involving the energy and time operators.

In order to construct unitary representations of T, we note that the set $\{I, P_\mu\}$ forms the basis of a maximal abelian subalgebra of $\mathfrak{T}$, and hence its elements may simultaneously be diagonalized:

$$I\,|\sigma p\rangle = \sigma\,|\sigma p\rangle$$

$$P_\mu\,|\sigma p\rangle = p_\mu\,|\sigma p\rangle$$

The identity operator is in fact a Casimir operator of T and so its eigenvalues serve to distinguish the different irreducible representations of T. Defining the inner product of two eigenvectors of I and P_μ to be

$$\langle\sigma' p'\,|\,\sigma p\rangle = \delta(\sigma' - \sigma)\,\delta(p' - p) \tag{28}$$

we obtain a generalized Hilbert space $\mathfrak{H}_P$ spanned by all $|\sigma p\rangle$, $-\infty < Re\,\sigma,\ p < \infty$, $|Im\,\sigma| \to 0$. Alternately, we may choose to diagonalize $\{I, X_\mu\}$:

$$I\,|\sigma x\rangle = \sigma\,|\sigma x\rangle$$

$$X_\mu\,|\sigma x\rangle = x_\mu\,|\sigma x\rangle$$

Now we have the Hilbert space $\mathfrak{H}_X \cong \mathfrak{H}_P$ spanned by the vectors $|\sigma x\rangle$ with the inner product

$$\langle \sigma' x' \,|\, \sigma x\rangle = \delta\,(\sigma' - \sigma)\,\delta\,(x' - x) \tag{29}$$

Other maximal abelian subalgebras are possible and will be discussed later.

Consider now the transformation properties of our new state vectors. It is trivial that

$$U(\alpha)\,|\sigma p\rangle = e^{-i\alpha\sigma}\,|\sigma p\rangle$$

$$U(a)\,|\sigma p\rangle = e^{-ia\cdot p}\,|\sigma p\rangle$$

$$U(\alpha)\,|\sigma x\rangle = e^{-i\alpha\sigma}\,|\sigma x\rangle$$

$$U(v)\,|\sigma x\rangle = e^{-iv\cdot x}\,|\sigma x\rangle$$

Now

$$\begin{aligned} P_\mu\,\{U(v)\,|\sigma p\rangle\} &= U(v)\,\{E(v)\,P_\mu\}\,|\sigma p\rangle \\ &= U(v)\,(P_\mu + v_\mu I)\,|\sigma p\rangle \\ &= (p_\mu + \sigma v_\mu)\,\{U(v)\,|\sigma p\rangle\} \end{aligned}$$

Also $I\,\{U(v)\,|\sigma p\rangle\} = \sigma\{U(v)\,|\sigma p\rangle\}$ so that we may write

$$U(v)\,|\sigma p\rangle = |\sigma\, p + \sigma v\rangle$$

setting the arbitrary phase equal to zero. Similarly, we obtain

$$U(a)\,|\sigma x\rangle = |\sigma\, x - \sigma a\rangle$$

To find the transformation coefficient between $|\sigma p\rangle$ and $|\sigma x\rangle$, we note the following string of equalities:

$$\begin{aligned} \langle \sigma' x\,|\,U(\alpha, v, a)\,|\sigma p\rangle &= e^{-i\alpha\sigma} e^{-ia\cdot p}\,\langle \sigma' x\,|\,\sigma\, p + \sigma v\rangle \\ &= (U(-a)\,U(-v)\,U(-\alpha)\,\phi\,(\sigma' x), \phi\,(\sigma p)) \\ &= e^{-i\alpha\sigma'} e^{-iv\cdot x}\,\langle \sigma\, x + \sigma' a\,|\,\sigma p\rangle \end{aligned} \tag{30}$$

We immediately see that the ansatz

$$\langle \sigma' x\,|\,\sigma p\rangle = (2\pi\sigma)^{-2}\,\delta\,(\sigma' - \sigma)\,e^{-ix\cdot p/\sigma} \tag{31}$$

satisfies the equality between the second and fourth expressions in Eq. (30). We have the orthogonality relation

$$\begin{aligned} \langle \sigma' p'\,|\,\sigma p\rangle &= \int d\sigma'' \int dx\,\langle \sigma' p'\,|\,\sigma'' x\rangle\,\langle \sigma'' x\,|\,\sigma p\rangle \\ &= \delta\,(\sigma' - \sigma)\,\delta\,(p' - p) \\ dx &= dx_0 dx_1 dx_2 dx_3 \end{aligned}$$

in agreement with Eq. (28). We shall identify the isomorphic Hilbert spaces $\mathfrak{H}_P$ and $\mathfrak{H}_X$ and simply write $\mathfrak{H}$. The sets $|\sigma p\rangle$ and $|\sigma x\rangle$ may then be regarded as just two different bases of $\mathfrak{H}$ related to each other by Eq. (31). We record the unitary transformation properties of the basis vectors:

$$\begin{aligned} U(\alpha, v, a)\,|\sigma p\rangle &= e^{-i\alpha\sigma} e^{-ia\cdot p}\,|\sigma\, p + \sigma v\rangle \\ U(\alpha, v, a)\,|\sigma x\rangle &= e^{-i\alpha\sigma} e^{-iv\cdot(x - \sigma a)}\,|\sigma\, x - \sigma a\rangle \end{aligned} \tag{32}$$

A concrete form of unitary representations of T is obtained by the following construction. Let us introduce the correspondence

$$|\sigma p\rangle \leftrightarrow \phi_{\sigma p}(x) = (2\pi\sigma)^{-2}\,e^{-ip\cdot x/\sigma} \equiv \langle x\,|\,p\rangle_\sigma$$

and the inner product

$$(\phi_{\sigma p}, \phi_{\sigma p'}) = \int dx\,\phi_{\sigma p}(x)^*\,\phi_{\sigma p'}(x) = \delta\,(p - p')$$

The set of all $\phi_{\sigma p}(x)$ for fixed σ forms the basis of the (generalized) Hilbert subspace $\mathfrak{H}_\sigma$ of $\mathfrak{H}$, irreducible under T. Since $\phi_{\sigma p}(x) = \phi_{-\sigma -p}(x)$, we see that irreducible representations of T characterized by σ and $-\sigma$ are equivalent. Hence it will suffice in the future to consider only the positive-σ representations.

The structure of L_0 is considerably more complicated than that of T. All the irreducible unitary representations L_0 are known (Ref. 46) and are classified by the eigenvalues of its two Casimir operators F and G. Writing

$$S_{\mu\nu} = (\mathbf{S}, \mathbf{T})$$

$$\mathbf{S} = (S_1, S_2, S_3) = (S_{23}, S_{31}, S_{12})$$

$$\mathbf{T} = (T_1, T_2, T_3) = (S_{01}, S_{02}, S_{03})$$

they can be expressed as follows:

$$F = -\frac{1}{2} S_{\mu\nu} S^{\mu\nu} = \mathbf{T}^2 - \mathbf{S}^2 \qquad (33)$$

$$G = \frac{1}{2} S_{\mu\nu} \widetilde{S}^{\mu\nu} = 2\mathbf{T} \cdot \mathbf{S} \qquad (34)$$

If we use the operators P_μ, W_μ, and Q_μ instead of $S_{\mu\nu}$, then we find the alternate covariant expressions

$$F = (W^2 - Q^2)/P^2$$
$$G = 2W \cdot Q/P^2$$

in every representation of the augmented Poincaré group in which P^2 is chosen to be diagonal and not equal to zero. We conventionally choose $\mathbf{S}^2$ and S_3 for the basis of a maximal abelian subalgebra of the enveloping algebra of $\mathfrak{L}_0$. Then we may introduce the vectors $|k\nu j\mu\rangle$ defined by the following eigenvalue equations:

$$(F, G, \mathbf{S}^2, S_3)|k\nu j\mu\rangle = (1 + \nu^2 - k^2, 2k\nu, j(j+1), \mu)|k\nu j\mu\rangle$$

Here

$$j = k, k+1, k+2, \cdots$$
$$\mu = j, j-1, \cdots, -j+1, -j$$

The numbers k and ν determine the following classes of irreducible unitary representations of L_0:

(i) $k = 0, \nu = i$

(ii) $k = 0, \nu \geqq 0$

(iii) $k = 0, \nu = i\nu_0, 0 < \nu_0 < 1$

(iv) $k = 1, 2, 3, \cdots, -\infty < \nu < \infty$

(v) $k = 1/2, 3/2, 5/2, \cdots, -\infty < \nu < \infty$

The representation (*i*) is the trivial or the identity representation. All representations are single-valued except for (*v*) which is double-valued.

The set $\{|k\nu j\mu\rangle\}$ (with the above restrictions on k, ν, j, and μ) forms a basis for the Hilbert space $\mathfrak{H}(L_0)$ of unitary representations of L_0 with the inner product

$$\langle k\nu j\mu | k'\nu' j'\mu'\rangle = \delta_{kk'}\, \delta(\nu - \nu')\, \delta_{jj'}\, \delta_{\mu\mu'}$$

Consider now unitary transformations of the basis vectors $|k\nu j\mu\rangle$. In view of the commutation relations

$$[S_i, S_j] = i e_{ijk} S_k$$
$$[S_i, T_j] = i e_{ijk} T_k$$
$$[T_i, T_j] = -i e_{ijk} S_k$$

the operators S_i span a subalgebra of $\mathfrak{L}_0$ isomorphic to the Lie algebra of the three-dimensional rotation group O_3^+. If $R \in O_3^+$, then evidently

$$U(R)|k\nu j\mu\rangle = \sum_{\mu'} D^j_{\mu'\mu}(R)|k\nu j\mu'\rangle$$

where the rotation matrices $D^j_{\mu'\mu}$ are given in Appendix D. In terms of Euler angles, we may write

$$U(R) = \exp(-i\alpha S_3)\exp(-i\beta S_2)\exp(-i\gamma S_3) \qquad (35)$$

Every "space-time" rotation Λ can be factored (Ref. 14 and 46) into a product of two spatial rotations and a pure Lorentz transformation in the z-direction:

$$\Lambda = R'ZR$$

with

$$U(R') = \exp(-i\alpha' S_3)\exp(-i\beta' S_2)$$
$$U(Z) = \exp(-i\zeta T_3)$$

and $U(R)$ is given by Eq. (35). The relation between the parameters $\alpha', \beta', \zeta', \alpha, \beta, \gamma$ and Λ^ν_μ is given in Appendix D. Noting that $[S_3 T_3] = 0$, we see that $U(Z)$ does not mix the μ-eigenvalues so that

$$U(Z)|k\nu j\mu\rangle = \sum_{j'} \mathfrak{Z}^{j\nu\mu}_{j'j}(\zeta)|k\nu j'\mu\rangle$$

where the $\mathfrak{Z}$-functions are also given in Appendix D. It follows that for an arbitrary Lorentz transformation $\Lambda \in L_0$ we have

$$U(\Lambda)|k\nu j\mu\rangle = \sum_{j'\mu'} Q^{k\nu}_{j'\mu' j\mu}(\Lambda)|k\nu j'\mu'\rangle$$
$$Q^{k\nu}_{j'\mu' j\mu}(\Lambda) = \sum_{\mu''} D^{j'}_{\mu'\mu''}(R')\, \mathfrak{Z}^{k\nu\mu''}_{j'j}(\zeta)\, D^j_{\mu''\mu}(R)$$

We are now in a position to construct irreducible unitary representations of P''. In view of Eq. (24) and the remarks immediately following this isomorphism, we have

$$P'' \cong T \times L_0$$

Introduce the Hilbert space $\mathfrak{H}(P'') = \mathfrak{H}(T) \times \mathfrak{H}(L_0)$ by exhibiting its basic vectors:

$$|\sigma k\nu p j\mu\rangle = |\sigma p\rangle \times |k\nu j\mu\rangle$$

The inner product in $\mathfrak{H}(P'')$ is by definition

$$\langle \sigma k\nu p j\mu | \sigma' k' \nu' p' j' \mu' \rangle = \langle \sigma p | \sigma' p' \rangle \langle k\nu j\mu | k'\nu' j'\mu' \rangle$$

The numbers σ, k, and ν label the different irreducible representations of P''. It should be clear from the preceding construction that we have found *all* such representations. An arbitrary element of P'' is the quadruplet (α, v, a, Λ) satisfying the group law

$$(\alpha', v', a', \Lambda')(\alpha, v, a, \Lambda) = (\alpha' + \alpha + a' \cdot v, v' + v, a' + a, \Lambda'\Lambda)$$

The unitary operators representing P'' are of the form

$$U(\alpha, v, a, \Lambda) = U(\alpha, v, a)\, U(\Lambda) = U(\Lambda)\, U(\alpha, v, a)$$

where

$$U(\Lambda) = \exp(-i\Omega{:}S/2)$$

$$\Lambda_{\mu\nu} = (e^{\Omega})_{\mu\nu} = g_{\mu\nu} + \Omega_{\mu\nu} + \frac{1}{2!} g^{\rho\sigma}\Omega_{\mu\rho}\Omega_{\sigma\nu} + \cdots$$

Using previous results, it is trivially found that

$$U(\alpha, v, a, \Lambda) |\sigma k\nu p j\mu\rangle = e^{-i\alpha\sigma} e^{-ia\cdot p} \sum_{j'\mu'} Q^{k\nu}_{j'\mu' j\mu}(\Lambda)\, |\sigma k\nu\, p + \sigma v\, j'\mu'\rangle$$

It should be clear that an irreducible unitary representation of P'' of the above form furnishes a unitary, although in general reducible, representation of P_0. The point is that $\mathfrak{P}_0$ is a Lie subalgebra of $\mathfrak{P}$ but not of $\mathfrak{P}''$ so that the vectors in $\mathfrak{H}(P'')$, whose construction is based on $\mathfrak{P}''$, will in general be mixed under the transformations of P_0 generated by the operators in $\mathfrak{P}_0$. What we obviously need is a different maximal abelian subalgebra suitable for the representations of P_0. In other words, we want the unitary operators of P to have the form

$$U(\alpha, v, a, l) = U(\alpha, v)\, U(a, l)$$

$$U(\alpha, v) = \exp(-i\alpha I)\exp(-iv \cdot X)$$

$$U(a, l) = \exp(-ia \cdot P)\exp(-i\omega{:}M/2)$$

clearly exhibiting the subgroup nature of the restricted Poincaré group. The augmented Poincaré group P is defined as the group of all quadruplets (α, v, a, l) with the group law

$$(\alpha', v', a', l')(\alpha, v, a, l) = (\alpha' + \alpha + a' \cdot l'v, v' + l'v, a' + l'a, l'l)$$

which may be worked out by considering the unitary representatives of these quadruplets. Just as T, the group P may be realized as the group of all 6×6 real matrices of the form[24]

$$\begin{bmatrix} 1 & \tilde{v}l & \alpha \\ 0 & l & a \\ 0 & 0 & 1 \end{bmatrix} \tag{36}$$

where

$$l = (l_\mu{}^\nu)$$

$$a = (a_\mu)$$

$$v = (v^\mu)$$

so that vl is a row vector with components $(\tilde{v}l)^\mu = v^\nu l_\nu{}^\mu$.

The vectors $|\sigma k\nu p j\mu\rangle$ are eigenvectors of the Casimir operators I, F, G and of P_μ, S^2, S^3 which form the basis of a maximal abelian subalgebra of $\mathfrak{E}$. Another such subalgebra is spanned by P_μ, W^2, and $W_0(\mathbf{P}^2)^{-1/2}$. Let us denote the eigenvectors of these operators by $|\sigma k\nu psh\rangle$; thus

$$(I, F, G, P_\mu, W^2, W_0(\mathbf{P}^2)^{-1/2})|\sigma k\nu psh\rangle = (\sigma, 1 + \nu^2 - k^2, 2k\nu, p_\mu, -m^2 s(s+1), h)|\sigma k\nu psh\rangle$$

Keeping σ, k, ν fixed and restricting the transformation of P to those of its subgroup P_0, we obviously obtain irreducible unitary representations of P_0. In fact, any vector of the form

$$|psh\rangle = \sum_{k \leq s} \int d\sigma \int d\nu\, w_{\sigma k\nu} |\sigma k\nu psh\rangle$$

where $w_{\sigma k\nu}$ is a complex function of its arguments, transforms irreducibly under P_0. The basis vectors $|\sigma k\nu psh\rangle$ span the representation Hilbert space $\mathfrak{H}(P)$ of P and are

[24] The author is indebted to Dr. M. M. Saffren for a discussion on matrix realizations of P.

related to the basis vectors $|\sigma k\nu p j\mu\rangle$ of $\mathfrak{H}(P'')$ by the unitary transformation

$$|\sigma k\nu psh\rangle = \sum_{j\mu} M^{k\nu}_{j\mu;sh}(p)\,|\sigma k\nu pj\mu\rangle$$

with the M-functions computed in Appendix F. In other words, $\mathfrak{H}(P)$ is isomorphic to $\mathfrak{H}(P'')$, and hence the representations of P'' are indeed representations of P. If $(\alpha, v, a, l) \in P$, then, according to the results of Section III and Appendix D, we have

$$U(\alpha, v, a, l)\,|\sigma k\nu psh\rangle = e^{-i\alpha\sigma} e^{-ia\cdot lp}\, U(v) \sum_{h''} D^{s}_{h''h}(R_w(a,l))\,|\sigma k\nu lpsh''\rangle$$

for the case of $p_0, p^2 > 0$. From Appendix F, on the other hand, we get for sufficiently small v

$$U(v)\,|\sigma k\nu lpsh''\rangle = \sum_{s'h'} \mathfrak{M}^{\sigma k\nu}_{s'h';sh''}(lp, v)\,|\sigma k\nu\, lp + \sigma v\, s'h'\rangle$$

with the $\mathfrak{M}$-functions discussed there. Putting everything together, we obtain

$$U(\alpha, v, a, l)\,|\sigma k\nu psh\rangle = e^{-i\alpha\sigma} e^{-ia\cdot lp} \sum_{s'h'} \left\{ \sum_{h''} D^{s}_{h''h}(R_w(a,l))\,\mathfrak{M}^{\sigma k\nu}_{s'h';sh''}(lp, v) \right\} \cdot |\sigma k\nu\, lp + \sigma v\, s'h'\rangle$$

This formula shows that the transformations of P indeed mix irreducible unitary representations of P_0 labeled by different spin values as announced in Section II.

Finally, we wish to consider the question of discrete automorphisms anew within the framework of the augmented Poincaré group. In order that the automorphisms σ, τ, and ρ of $\mathfrak{P}_0$, given by Eq. (12), (13), and (14), be those of $\mathfrak{P}$, the X_μ must transform in the same way as the P_μ, and I must stay unchanged:

$$\sigma\!: \; X_\mu \to {}^{\sigma}X_\mu = X^\mu = (X_0, -\mathbf{X})$$

$$\tau\!: \; X_\mu \to {}^{\tau}X_\mu = -X^\mu = (-X_0, \mathbf{X})$$

$$\rho\!: \; X_\mu \to {}^{\rho}X_\mu = -X_\mu = (-X_0, -\mathbf{X})$$

This set of transformations is of course consistent with the duality between P_μ and X_μ. The anti-automorphism γ, given by Eq. (15), must transform the i in $[P_\mu, X_\nu] = ig_{\mu\nu}I$ into $-i$; this can be accomplished either by having $X_\nu \to X_\nu$, $I \to I$ or $X_\nu \to -X_\nu$, $I \to -I$. We choose the second alternative in order to maintain the duality:

$$\gamma\!: \; P_\mu \to -P_\mu, X_\mu \to -X_\mu, I \to -I$$

In addition to the above symmetries of $\mathfrak{P}$, we have further discrete automorphisms and anti-automorphisms generated by the "duality-breaking" automorphism δ of $\mathfrak{X}$:

$$\delta\!: \; P_\mu \to P_\mu, X_\mu \to -X_\mu, I \to -I$$

Let $V' = \{\omega' : \omega' = \delta\omega, \omega \in V\}$. Then the set $W = V \cup V'$ is an abelian group of order 8 with the multiplication table

$\cdot$	V	V'
V	V	V'
V'	V'	V

Here VV', e.g., is the set of elements of the form $\omega\omega'$ with $\omega \in V$ and $\omega' \in V'$. Since $w^{-1}Vw \in V$ for an arbitrary $w \in W$, it follows that V is a normal subgroup of W. It should be clear that W is a group of discrete automorphisms not only of $\mathfrak{X}$ but also of $\mathfrak{P}$. Moreover, every anti-automorphism of $\mathfrak{P}$ has the form $\gamma \cdot w$, $w \in W$.

V. TENSOR PRODUCT REPRESENTATIONS

This section is primarily of a mathematical character. Our principal aim here is to present the rudiments of the group representation theory on tensor product spaces. We shall use the basic group T as an example; the treatment may easily be adapted to other groups.[25]

We shall be dealing with the basic state vectors $|\sigma p\rangle$ for which we introduce the abbreviated notation $\phi_\lambda, \lambda = (\sigma, p)$. The (generalized) Hilbert space $\mathfrak{H}$ introduced in Section IV is spanned by these (singular) basis vectors. We write

$$(\phi_\lambda, \phi_{\lambda'}) = \delta_{\lambda\lambda'} \tag{37}$$

as a shorthand for Eq. (28). Every element $\psi \in \mathfrak{H}$ may be expressed in the form

$$\psi = \int d\lambda c_\lambda \phi_\lambda$$

where $\int d\lambda = \int d\sigma \int dp$ and the c_λ are complex numbers or distributions. We recall that ψ is said to be regular whenever

$$\|\psi\|^2 = \int d\lambda |c_\lambda|^2 < \infty$$

Let ψ and χ be two elements of $\mathfrak{H}$. The tensor product of ψ and χ, denoted by $\psi \otimes \chi$, is a mapping of the pair ψ, χ into a linear vector space V. By definition, the tensor product is linear in each of its factors:

$$\begin{aligned} \alpha(\psi \otimes \chi) = (\alpha\psi) \otimes \chi = \psi \otimes (\alpha\chi), \ \alpha \text{ complex} \\ \psi \otimes (\chi_1 + \chi_2) = \psi \otimes \chi_1 + \psi \otimes \chi_2 \\ (\psi_1 + \psi_2) \otimes \chi = \psi_1 \otimes \chi + \psi_2 \otimes \chi \end{aligned}$$

If we introduce the inner product

$$(\psi \otimes \chi, \psi' \otimes \chi') = (\psi, \psi')(\chi, \chi')$$

then the linear closure of V is a Hilbert space, the tensor product of $\mathfrak{H}$ with itself, denoted by $\mathfrak{H} \otimes \mathfrak{H} = \mathfrak{H}^{(2)}$. The basis of $\mathfrak{H}^{(2)}$ consists of all tensor products $\phi_{\lambda_1} \otimes \phi_{\lambda_2}$, where ϕ_{λ_1} and ϕ_{λ_2} are basis elements of $\mathfrak{H}$.

[25]For a rigorous mathematical discussion of topics related to this section see Ref. 47 and 48.

We may generalize the foregoing by defining higher order tensor products. Thus for an arbitrary integer $n \geqq 1$,

$$\mathfrak{H}^{(n)} = \mathfrak{H} \otimes \cdots \otimes \mathfrak{H} \qquad (n \text{ times})$$

has for its basis the vectors

$$\phi_{\lambda_1 \cdots \lambda_n} = \phi_{\lambda_1} \otimes \cdots \otimes \phi_{\lambda_n}$$

where the ϕ_{λ_k} are the basis vectors of $\mathfrak{H}$; here $\mathfrak{H}^{(1)} = \mathfrak{H}$. The inner product in $\mathfrak{H}^{(n)}$ is by definition

$$(\psi_1 \otimes \cdots \otimes \psi_n, \chi_1 \otimes \cdots \otimes \chi_n) = \prod_{k=1}^{n} (\psi_k, \chi_k)$$

The most general element in $\mathfrak{H}^{(n)}$ is of the form

$$\psi = \int d\lambda_1 \cdots \int d\lambda_n c_{\lambda_1 \cdots \lambda_n} \phi_{\lambda_1 \cdots \lambda_n}$$

The vector ψ is regular if and only if

$$\|\psi\|^2 = \int d\lambda_1 \cdots \int d\lambda_n |c_{\lambda_1 \cdots \lambda_n}|^2 < \infty$$

where we have used Eq. (37).

Let us consider operators on $\mathfrak{H}^{(n)}$ for some fixed $n \geqq 1$. Given the set of operators $\{A_1, A_2, \cdots, A_n\}$ on $\mathfrak{H}$ to $\mathfrak{H}$, we define the tensor product operator $A_1 \otimes \cdots \otimes A_n$ on $\mathfrak{H}^{(n)}$ to $\mathfrak{H}^{(n)}$ by

$$(A_1 \otimes \cdots \otimes A_n)(\psi_1 \otimes \cdots \otimes \psi_n) = (A_1\psi_1) \otimes \cdots \otimes (A_n\psi_n)$$

for each $\psi_1 \otimes \cdots \otimes \psi_n \in \mathfrak{H}^{(n)}$ for which the right-hand side above is defined. It is easy to verify that $A_1 \otimes \cdots \otimes A_n$ is a linear operator whenever each $A_k, k = 1, \cdots, n$, is. Of particular interest to us are the operators

$$A^{(n)}(k) = 1 \otimes \cdots \otimes 1 \otimes A \otimes 1 \otimes \cdots \otimes 1, \qquad A \in \mathfrak{T}$$

where A is in the kth place and each of the $n-1$ 1's is the identity operator on $\mathfrak{H}$ leaving each vector of $\mathfrak{H}$ fixed. The set of all $A^{(n)}(k)$, $A \in \mathfrak{T}$, forms the Lie algebra $\mathfrak{T}^{(n)}(k)$ isomorphic to $\mathfrak{T}$. It is easy to see that $\mathfrak{T}^{(n)}(k)$ is orthogonal to $\mathfrak{T}^{(n)}(l)$ whenever $k \neq l$; i.e., each element of $\mathfrak{T}^{(n)}(k)$ commutes with every element of $\mathfrak{T}^{(n)}(l)$. For the basis of $\mathfrak{T}^{(n)}(k)$ we have the set

$$\mathfrak{B}^{(n)}(k) = \{I^{(n)}(k), P_\mu^{(n)}(k), X_\mu^{(n)}(k)\}$$

The Lie algebra spanned by the operators in

$$\mathfrak{B}_{\text{ext}}^{(n)} = \{A^{(n)} = \sum_{k=1}^{n} A^{(n)}(k) \colon A^{(n)}(k) \in \mathfrak{B}^{(n)}(k)\}$$

is called the external Lie algebra $\mathfrak{T}_{\text{ext}}^{(n)}$ of $\mathfrak{H}^{(n)}$.

Unitary representations of T on $\mathfrak{H}^{(n)}$ are generated by the basis elements of $\mathfrak{T}_{\text{ext}}^{(n)}$. Thus, e.g.,

$$\begin{aligned} U^{(n)}(a) &= \exp[-ia \cdot P^{(n)}] \\ &= \exp\left[-ia \cdot \sum_{l=\gamma} P^{(n)}(k)\right] \\ &= \exp[-ia \cdot P^{(n)}(1)] \cdots \exp[-ia \cdot P^{(n)}(n)] \end{aligned}$$

with the interpretation

$$[P^{(n)}(k)]^0 = 1 \otimes \cdots \otimes 1 \qquad (n \text{ times})$$

For an arbitrary $\psi_1 \otimes \cdots \otimes \psi_n \in \mathfrak{H}^{(n)}$, we then have

$$\begin{aligned} &U^{(n)}(\alpha, v, a)(\psi_1 \otimes \cdots \otimes \psi_n) \\ &\qquad = \{U(\alpha, v, a)\psi_1\} \otimes \cdots \otimes \{U(\alpha, v, a)\psi_n\} \end{aligned}$$

Suppose we take the basis vector $\phi_{\lambda_1 \cdots \lambda_n} \in \mathfrak{H}^{(n)}$. Then each factor $\phi_{\lambda_k}(\lambda_k = (\sigma_k, p_k))$ transforms irreducibly under T. But so also does the tensor product $\phi_{\lambda_1 \cdots \lambda_n}$ because it is an eigenvector of $I^{(n)}$ and $P_\mu^{(n)}$ with the eigenvalues $\sigma_1 + \cdots + \sigma_n$ and $p_{1\mu} + \cdots + p_{n\mu}$. That is to say, tensor product representations of irreducible unitary representations of T are themselves irreducible. This is a very special and fortunate property of the group T and is of course a consequence of the additivity of I's and P's.

By its construction, each $\mathfrak{H}^{(n)}$ is closed under superposition. This is not true for the operation of composition, however. In fact, the tensor product of a vector $\psi^{(m)} \in \mathfrak{H}^{(m)}$ and $\chi^{(n)} \in \mathfrak{H}^{(n)}$ lies in $\mathfrak{H}^{(m \times n)}$. As we have mentioned in Section II, this undesirable lack of closure may be remedied by introducing an infinite-fold tensor product Hilbert space $\mathfrak{H}^\infty$. We define it as follows. For $m \neq n$, the inner product of any two vectors $\psi^{(m)} \in \mathfrak{H}^{(m)}$ and $\chi^{(n)} \in \mathfrak{H}^{(n)}$ is by definition equal to zero. The spaces $\mathfrak{H}^{(n)}$ for different n are thus mutually orthogonal, and one may form their direct sum:

$$\mathfrak{H}^\infty = \sum_{n=1}^{\infty} \otimes \mathfrak{H}^{(n)}$$

An arbitrary vector $\psi \in \mathfrak{H}^\infty$ may be written uniquely as

$$\psi = \sum_{n=1}^{\infty} \psi^{(n)}, \psi^{(n)} \in \mathfrak{H}^{(n)}$$

Thus

$$(\psi, \chi) = \sum_{n=1}^{\infty} (\psi^{(n)}, \chi^{(n)})$$

for any two vectors $\psi, \chi \in \mathfrak{H}^\infty$. A vector $\psi \in \mathfrak{H}^\infty$ is regular whenever $\|\psi\| < \infty$. But this means that each $\psi^{(n)}$ is regular and, moreover, that the series $\Sigma_n \|\psi^{(n)}\|^2$ converges. The regularity of each $\psi^{(n)}$ is not sufficient to guarantee that of ψ.

Operators on $\mathfrak{H}^\infty$ are defined in a manner analogous to that for $\mathfrak{H}^{(n)}$. Thus, e.g., we shall write

$$P_\mu(k) \equiv P_\mu^\infty(k) = 1 \otimes \cdots \otimes 1 \otimes P_\mu \otimes 1 \otimes \cdots$$

with P_μ in the kth place. Also,

$$P_\mu \equiv P_\mu^\infty = \sum_{k=1}^{\infty} P_\mu(k)$$

etc. It should be clear that an operator on $\mathfrak{H}^{(n)}$ may be extended to one on $\mathfrak{H}^\infty$ by simply post-multiplying it by the identity operator $1 \equiv 1^\infty = 1 \otimes 1 \otimes \cdots$ on $\mathfrak{H}^\infty$:

$$A^{(n)} \to A^{(n)} \otimes 1 \otimes 1 \otimes \cdots$$

From now on we shall always deal with $\mathfrak{H}^\infty$ and shall regard each $\mathfrak{H}^{(n)}$ as a subspace of $\mathfrak{H}^\infty$ containing vectors of the form

$$\psi = (\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n) \otimes \phi_{(0,0)} \otimes \phi_{(0,0)} \otimes \cdots$$

for some $\psi_i \in \mathfrak{H}^{(1)}$.

As we shall explain later, the state of any physical system may be represented by a vector in $\mathfrak{H}^\infty$. It behooves us therefore to examine the structure of $\mathfrak{H}^\infty$ in some detail. Let us write

$$\phi_{\sigma_k} \equiv \phi_{(\sigma_k, 0)} = |\sigma_k p_k = 0\rangle$$
$$\phi_\sigma = \phi_{\sigma_1 \sigma_2 \cdots} = \phi_{\sigma_1} \otimes \phi_{\sigma_2} \otimes \cdots$$

The vector ϕ_σ is clearly in $\mathfrak{H}^\infty$. Applying $P_\mu(k)$, we find for each $k = 1, 2, \cdots$

$$P_\mu(k)\,\phi_\sigma = 0$$

and hence

$$\sum_{k=1}^{\infty} P_\mu(k)\,\phi_\sigma = P_\mu \phi_\sigma = 0$$

Thus ϕ_σ is a state of zero total linear momentum. As we shall show in Section VI, the total angular momentum operator on $\mathfrak{H}^\infty$ is

$$M_{\mu\nu} = \sum_{k=1}^{\infty} \frac{X_{[\mu}(k)\,P_{\nu]}(k)}{I(k)}$$

Operating with $M_{\mu\nu}$ on ϕ_σ we see that it too gives a zero result (since the $P_\mu(k)$ annihilate ϕ_σ). In other words, ϕ_σ has the Poincaré-invariant properties associated with a physical vacuum:

$$U(a, l)\,\phi_\sigma = \phi_\sigma$$

We shall tentatively assume that ϕ_σ indeed represents a physical vacuum state. Note that with this interpretation the physical vacuum is not unique because it is described by ϕ_σ for any sequence $\sigma = (\sigma_1, \sigma_2, \cdots)$ provided the σ_k are not all zero.

Now we show that every vector in $\mathfrak{H}^\infty$ may be obtained from ϕ_σ with various σ's. Consider the operator

$$0_p = \exp\left[-i \sum_{k=1}^{\infty} p_k \cdot X(k)/I(k)\right]$$
$$= \prod_{k=1}^{\infty} \exp\left[-p_k \cdot X(k)/I(k)\right]$$

Clearly,

$$0_p \phi_\sigma = \phi_{\lambda_1} \otimes \phi_{\lambda_2} \otimes \cdots$$
$$\lambda_k = (\sigma_k, p_k)$$

But the set of all $0_p \phi_\sigma$ is a basis of $\mathfrak{H}^\infty$; hence follows the truth of the above assertion. We see that the application of an appropriate operator 0_p to the vacuum state vector ϕ_σ describes mathematically the "excitation" of the vacuum into a state of nonzero linear (and hence angular) momenta. We may therefore interpret 0_p as a creation operator of "particles" with various momenta $p_1, p_2, \cdots$. In particular, $e^{-ip\cdot X/I}$ is the creation operator of a basic particle of momentum p:

$$e^{-ip\cdot X/I}|\sigma 0\rangle = |\sigma p\rangle$$

It should hardly be necessary to emphasize that the above creation operators have nothing to do with those of bare or of physical particles encountered in field theory. One of their peculiar properties, e.g., is that their adjoints do not annihilate the vacuum but create particles of opposite momentum:

$$(e^{-ip\cdot X/I})^* |\sigma 0\rangle = |\sigma - p\rangle$$

VI. INTERNAL SYMMETRIES

This section marks our return to more physical matters. Starting with the simplest cases, we shall construct various state vectors, in the order of increasing complexity, always being careful to provide as much physical motivation and interpretation as possible. We shall find that the representation theory of our basic group T permits us to construct systematically state vectors characterized by quantum numbers such as spin, isospin, baryon number, etc. Moreover, we shall obtain an infinite hierarchy of internal symmetry groups according to which the various particles occurring in nature may be classified. We emphasize that what we find is the set of all *possible* one-particle states; the physically observed states form only a small subset of these and are determined by the much more difficult dynamical considerations to appear elsewhere.

Let us start our discussion by considering the simplest of all possible state vectors, namely the basic vectors $|\sigma p\rangle$. As we have already explained in Section IV, for fixed σ these vectors form a basis for an irreducible unitary representation of our basic group T. Physically, the vectors $|\sigma p\rangle$ describe a system of given four-momentum p (which may be timelike, lightlike, or spacelike) with all other quantum numbers suppressed, ignored, or unknown. The eigenvalue σ specifies how the state vector behaves under the (unitary) phase transformations $\exp(-i\alpha I)$; the precise physical significance of σ can only be understood in a dynamical context. It may be appropriate, however, to point out that we expect that σ should be the same (equal to 1) for all physical particles in order that $[P_\mu, X_\nu] = ig_{\mu\nu}I\,(\hbar = 1)$ reduce to the canonical commutation relations for $\mu, \nu = 1, 2, 3$. The Lie algebra associated with these basic vector representations is of course spanned by I, P_μ, and X_μ. Let us define operators $L_{\mu\nu}$ belonging to the enveloping algebra $\mathfrak{E}$ of $\mathfrak{T}$:

$$L_{\mu\nu} = X_{[\mu}P_{\nu]}/I$$

Evidently, $L_{\mu\nu}$ is just the *orbital* angular momentum operator of this "one-particle" system; moreover, it coincides with the *total* angular momentum operator $M_{\mu\nu}$ since there are no other angular momentum operators to be constructed out of P_μ, X_μ, and I. If we adjoin $L_{\mu\nu} = M_{\mu\nu}$ to the basis set $\{I, P_\mu, X_\mu\}$, then we get a new set of operators forming a basis for the Lie algebra $\mathfrak{P}$ of the augmented Poincaré group P. This set yields only the trivial representation of L_0; the reason for this is of course that $S_{\mu\nu} = M_{\mu\nu} - L_{\mu\nu} = 0$. Incidentally, the enveloping algebra of $\mathfrak{P}$ is just that of $\mathfrak{T}$, namely $\mathfrak{E}$.

We now turn to the more interesting case of tensor products of two basic vectors: $|\sigma_1 p_1\rangle \otimes |\sigma_2 p_2\rangle$. These vectors span the representation space of the "two-particle" Lie algebra $\mathfrak{T}^{(2)} = \mathfrak{T}(1) \otimes \mathfrak{T}(2)$. We now have available the 18 operators $I(i)$, $P_\mu(i)$, and $X_\mu(i)$, $i = 1, 2$. The enveloping algebra $\mathfrak{E}^{(2)}$ of $\mathfrak{T}^{(2)}$ will contain not only operators in $\mathfrak{E}(1)$ and $\mathfrak{E}(2)$, the enveloping algebras of $\mathfrak{T}(1)$ and $\mathfrak{T}(2)$, but also operators which are mixtures of operators from $\mathfrak{T}(1)$ and $\mathfrak{T}(2)$. Clearly, the variety of interesting operators is now much richer than in the single-particle case analyzed above. Of a particular interest to us are the external operators in $\mathfrak{E}^{(2)}$:

$$I = I(1) + I(2)$$

$$P_\mu = P_\mu(1) + P_\mu(2)$$

$$X_\mu = X_\mu(1) + X_\mu(2)$$

As we have already explained, these operators are associated with the overall or "bulk" properties of the two-particle system. Clearly, P_μ is just the total linear four-momentum of the system, while $X_\mu/2$ is just the average four-position vector[23] of the two particles. The external operators form a Lie algebra, $\mathfrak{T}^{(2)}_{\text{ext}}$, which is isomorphic to $\mathfrak{T}^{(2)}$. In fact, all the Lie algebras $\mathfrak{T}$ (with various appendages) occurring in our theory will be isomorphic to each other and to the basic Lie algebra $\mathfrak{T}$; these isomorphisms will henceforth be taken for granted.

How are we to define the angular momentum operators for the case of two particles? We already have $M_{\mu\nu}(i) = X_{[\mu}(i)\,P_{\nu]}(i)/I(i)$, $i = 1, 2$, for each of the two particles. Now we appeal to our physical experience and *define* the total angular momentum of the two-particle system to be the sum of the total (in this case equal to the orbital) angular momenta of the individual particles:

$$M_{\mu\nu} = M_{\mu\nu}(1) + M_{\mu\nu}(2) \tag{38}$$

The orbital angular momentum of the two-particle system is of course

$$\begin{aligned} L_{\mu\nu} &= X_{[\mu}P_{\nu]}/I \\ &= [X_{[\mu}(1) + X_{[\mu}(2)]\,[P_{\nu]}(1) + P_{\nu]}(2)]\,[I(1) + I(2)]^{-1} \end{aligned}$$

Let us add $L_{\mu\nu}$ to the right-hand side of Eq. (38) and then subtract it. The result is

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$$

$$S_{\mu\nu} = \bar{X}_{[\mu}\bar{P}_{\nu]}/\bar{I} \tag{39}$$

where

$$\bar{I} = I(1)\,I(2)\,[I(1) + I(2)]$$

$$\bar{P}_\mu = P_\mu(1)\,I(2) - P_\mu(2)\,I(1)$$

$$\bar{X}_\mu = X_\mu(1)\,I(2) - X_\mu(2)\,I(1) \tag{40}$$

We call the Lie algebra spanned by $\bar{I}$, $\bar{P}_\mu$, and $\bar{X}_\mu$ the internal Lie algebra $\mathfrak{T}^{(2)}_{\text{int}}$ of the two-particle system. Definitions of internal operators are of course not unique; they are arbitrary to the extent that $\bar{I}$, $\bar{P}_\mu$, and $\bar{X}_\mu$ may be multiplied by various functions of $I(1)$ and $I(2)$ subject only to the conditions $\bar{X}_{[\mu}\bar{P}_{\nu]}/\bar{I} = S_{\mu\nu}$ and $[\bar{P}_\mu, \bar{X}_\nu] = ig_{\mu\nu}\bar{I}$. The reason for choosing the particular form of $\bar{P}_\mu$ is its simplicity. Equation (39) is interesting because it shows that the total angular momentum operator of a two-particle system contains both orbital and spin contributions, the latter arising from internal degrees of freedom of this system. The operators $S_{\mu\nu}$ of course satisfy the commutation relations (Eq. 23). The internal four-momentum $\bar{P}$ is essentially (ignoring the I's) the relative four-momentum between the two particles; $2\bar{X}$ is, again essentially, the four-position of particle 1 from the average position of the two particles.

The Lie algebras $\mathfrak{T}^{(2)}_{\text{ext}}$ and $\mathfrak{T}^{(2)}_{\text{int}}$ are easily seen to be orthogonal. This means that $L_{\mu\nu}$ and $S_{\mu\nu}$ commute, as of course they should. By combining the operators of $\mathfrak{T}^{(2)}_{\text{ext}}$ with the operators $S_{\mu\nu}$ from the enveloping algebra of $\mathfrak{T}^{(2)}_{\text{int}}$, we obtain a basis for the Lie algebra $\mathfrak{P}$ of P. Now we are able to obtain non-trivial representations of the Lorentz group L_0 in P. However, representations with $k_\nu = 0$ only can be secured because $S_{\mu\nu}\tilde{S}^{\mu\nu} = 0$; thus we have still not reached the most general case of spin angular momentum.

By the procedure outlined above, we have constructed the two mutually orthogonal Lie algebras $\mathfrak{T}^{(2)}_{\text{ext}}$ and $\mathfrak{T}^{(2)}_{\text{int}}$ whose operators respectively generate external and internal transformations of two-particle state vectors. One may inquire whether the operators of these two algebras can replace those of $\mathfrak{T}(1)$ and $\mathfrak{T}(2)$, i.e., whether the enveloping algebra $\mathfrak{E}'$ of $\mathfrak{T}^{(2)}_{\text{ext}} \oplus \mathfrak{T}^{(2)}_{\text{int}}$ is the same as $\mathfrak{E}$, that of $\mathfrak{T}(1) \oplus \mathfrak{T}(2)$. It turns out that this is not true. The reason is not hard to see. Given $I = I(1) + I(2)$ and $\bar{I} = I(1)I(2)[I(1) + I(2)]$, we cannot uniquely obtain $I(1)$ and $I(2)$ since both I and $\bar{I}$ are symmetric in $I(1)$ and $I(2)$. In order to remedy this situation, let us introduce the operator

$$\bar{I}' = I(1) - I(2)$$

Then

$$\bar{I} = (I^2 - \bar{I}'^2)\,I/4$$

The two sets of operators $\{I, P_\mu, X_\mu, \bar{I}', \bar{P}_\mu, \bar{X}_\mu\}$ and $\{I(i), P_\mu(i), X_\mu(i)\colon i = 1, 2\}$ are now equivalent, i.e., the generators of one set are uniquely expressible in terms of the other, and hence both sets generate identical enveloping algebras. Note that $\bar{I}'$ is *not* the commutator of $\bar{P}$ and $\bar{X}$; this fact will not create any difficulties.

Instead of the operators $I(i)$ and $P_\mu(i)$, $i = 1, 2$, we may diagonalize I, P_μ, $\bar{I}'$, and $\bar{P}_\mu$ and hence introduce the state vectors $|\sigma p; \tau p\rangle$ $(\tau = \sigma_1 - \sigma_2)$ which are eigenvectors of these ten operators. We impose the standard normalization on the new vectors:

$$\langle \sigma p; \tau\bar{p} \,|\, \sigma' p'; \tau'\bar{p}'\rangle = \delta(\sigma - \sigma')\,\delta(p - p')\,\delta(\tau - \tau')\,\delta(\bar{p} - \bar{p}')$$

By an elementary manipulation of the delta functions in this expression, one finds

$$\langle \sigma p; \tau\bar{p} \,|\, \sigma' p'; \tau'\bar{p}'\rangle = |2\sigma^4|^{-1} \langle \sigma_1 p_1 \,|\, \sigma_1' p_1'\rangle \langle \sigma_2 p_2 \,|\, \sigma_2' p_2'\rangle$$

i.e.,

$$|\sigma_1 p_1\rangle \otimes |\sigma_2 p_2\rangle = \sqrt{2}\,\sigma^2\,|\sigma p; \tau\bar{p}\rangle \tag{41}$$

if we choose the arbitrary phase factor to be unity. Equation (41) effects the reduction of the tensor product vector $|\sigma_1 p_1\rangle \otimes |\sigma_2 p_2\rangle$ to a vector irreducible under T; we see that this reduction is trivial in the sense that this vector already transforms irreducibly under T although the notation does not show it.

The next step in our program of state vector construction is to introduce eigenvectors of various spin operators. As we have seen in Section IV, we may simultaneously diagonalize the following four operators:

$$F = -\frac{1}{2} S_{\mu\nu}S^{\mu\nu} = 1 + \nu^2 - k^2$$

$$G = \frac{1}{2} S_{\mu\nu}\tilde{S}^{\mu\nu} = 2k\nu$$

$$\mathbf{S}^2 = s(s + 1)$$

$$S_3 = \mu$$

Of these, G vanishes. Thus we are left with the three operators F, $\mathbf{S}^2$, and S_3 instead of the four P_μ; it may appear that we may not be able to establish a one-to-one correspondence between the spin eigenvectors and the vectors $|\sigma p; \tau\bar{p}\rangle$. However, we recognize immediately that the operators $\bar{P}^2$, $\bar{P} \cdot \bar{X}$, and $\bar{X}^2$ commute with F, $\mathbf{S}^2$, and S_3 and hence are candidates for diagonalization. Only one of the three operators may be chosen to be diagonal, since they do not commute. For future convenience, we wish to introduce certain linear combinations of these operators. Let

$$A^{\pm}_\mu = (l_0\bar{P}_\mu \pm i\bar{X}_\mu/l_0)\,(2\bar{I})^{-1/2} \tag{42}$$

where l_0 is a constant of dimensions length or inverse mass; it may be regarded as a universal constant (the fundamental length) of our theory to be used in making certain dimensional expressions dimensionless. (It is interesting to note that with $\hbar$ and l_0 as fundamental constants both the product and the ratio of P and X are fixed: $PX \sim \hbar$, $P/X \sim \hbar/l_0^2$.) We shall henceforth choose our units so that $l_0 = 1$. The conversion factor or the value of l_0 in centimeters is to be determined by comparing future dynamical calculations with experiment. We find

$$[A_\mu^+, A_\nu^-] = g_{\mu\nu}$$
$$[A_\mu^\pm, A_\nu^\pm] = 0$$
$$(A_\mu^+)^* = A_\mu^-$$

We may note that the operators ξ^1, A_ν^+, and A^- satisfy commutation relations of the operator algebra of a linear harmonic oscillator. However, there is no lower bound to the eigenvalues of ξ_1^1 since

$$2(\psi, \xi_1^1\psi) = (\psi, A^+ \cdot A^-\psi) + (\psi, A^- \cdot A^+\psi)$$
$$= (A_\mu^- \psi, A^{-\mu}\psi) + (A_\mu^+ \psi, A^{+\mu}\psi)$$

is of indefinite sign, i.e., ξ^1 is not positive definite. Now we introduce the following Lorentz scalars or invariants:

$$\xi^{11} = A^+ \cdot A^+$$
$$\xi_{11} = A^- \cdot A^-$$
$$\xi_1^1 = A^+ \circ A^- \equiv \frac{1}{2}(A^+ \cdot A^- + A^- \cdot A^+)$$

Then

$$[\xi^{11}, \xi_{11}] = 4\xi^1$$
$$[\xi^1, \xi^{11}] = -2\xi^{11}$$
$$[\xi^1, \xi_{11}] = 2\xi_{11}$$

The hermitian operators

$$K_1 = (\xi^{11} + \xi_{11})/4$$
$$K_2 = i(\xi^{11} - \xi_{11})/4$$
$$K_3 = \xi_1^1/2 \tag{43}$$

satisfy commutation relations which are recognized as those of the generators of the 3-dimensional restricted Lorentz group $L_0^{(3)}$. For a discussion of the representation theory of this locally compact Lie group (Ref. 15) we refer the reader to Bargmann's classic paper (Ref. 49). Here we shall be content with the following observations. The group $L^{(3)}$ contains a one-parameter compact subgroup generated by $K_3 = (\overline{P}^2 + \overline{X}^2)/4\overline{I}$. Irreducible unitary representations of $L_0^{(3)}$ may be labeled by the eigenvalues q (discrete or continuous) and κ (discrete) of its Casimir operator $Q = K_1^2 + K_2^2 - K_3^2$ and the operator K_3, respectively. The irreducible unitary representations of $L_0^{(3)}$ and of $L_0^{(4)} \equiv L_0$, the 4-dimensional Lorentz group generated by the operators $S_{\mu\nu}$, are closely connected in virtue of the relation $F = 4Q$, as we shall show later. Thus one may write down the formal expansion

$$|\sigma p; \tau\overline{p}\rangle = \int df \sum_{\kappa, s, \mu} |\sigma p; \tau f\kappa s\mu\rangle \langle f\kappa s\mu|\overline{p}\rangle_{\overline{\sigma}}$$

where

$$\sigma = \sigma_1\sigma_2(\sigma_1 + \sigma_2) = (\sigma^2 - \tau^2)\sigma/4$$

The quantum numbers p, s, and μ have the simple physical interpretation of linear momentum, spin, and spin projection, respectively. The interpretation of κ must be deferred until we investigate higher order tensor products. No simple physical interpretation of f appears available. However, one can show that state vectors of stable physical systems are not eigenvectors[26] of F but are their mixtures. The reason for this is briefly as follows. Writing

$$\langle\sigma' x; \tau'\overline{x}|\sigma p; \tau f\kappa s\mu\rangle = \delta(\sigma - \sigma')\,\delta(\tau - \tau')\,\langle x|p\rangle_\sigma \langle\overline{x}|f\kappa s\mu\rangle_{\overline{\sigma}}$$

we may interpret $\langle x|p\rangle_\sigma = (2\pi\sigma)^{-2} e^{-ip\cdot x/\sigma}$ as the c.m. part of a wave function of a composite particle made up of two basic particles. The internal part of the wave function is just $\langle\overline{x}|f\kappa s\mu\rangle_{\overline{\sigma}}$. The trouble with this quantity is that, as one may show, it is not square integrable over the whole of $\overline{x}$-space, except for discrete values of f. Hence it does not represent a physical particle in the usual sense. However, the integral

$$\int df w(f) \langle\overline{x}|f\kappa s\mu\rangle_{\overline{\sigma}}$$

can be made square integrable by choosing a suitable weight function $w(f)$ and hence may represent the internal wave function of a composite particle.

Consider now the Lie algebra $\mathfrak{P}_{\text{ext}}^{(2)}$ of the augmented Poincaré group P spanned by the two-particle external operators I, P_μ, X_μ, and $M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$. These operators generate external unitary transformations which, acting on the states $|\sigma p; \tau f\kappa s\mu\rangle$, mix p, s, and μ (note that s and μ are *fixed* under T_{ext}!); hence these three quantum numbers are external under P. The remaining quantum numbers σ, τ, f, and κ are unchanged under all transformations of P and are thus internal (σ and τ are simultaneously external). In other words, the property of being an external or internal quantum number is relative;

[26]Except possibly for discrete eigenvalues $f = 1 - k^2$, $k = 0$, ½, 1, · · · .

it only makes sense if we specify the group of external transformations. To avoid possible misunderstanding, we shall occasionally indicate the external group in question by a prefix. Thus, in the case under consideration, the P-internal Lie algebra $\mathfrak{P}^{(2)}_{\text{int}}$ is just $\{I'\} \oplus \mathfrak{L}^{(3)}_{0}$.

We now proceed to generalize our discussion to the case of $(n+1)$-fold tensor products of the basic state vectors for an arbitrary positive integer n. We start with the Lie algebra

$$\mathfrak{T}^{(n+1)} = \mathfrak{T}(1) \otimes \cdots \otimes \mathfrak{T}(n+1)$$

spanned by the operators $I(i)$, $P_\mu(i)$ and $X_\mu(i)$, $i = 1, 2, \cdots, n+1$. The external Lie algebra $\mathfrak{T}^{(n+1)}_{\text{ext}}$ is of course spanned by

$$\begin{aligned} I^{(n+1)} &= \textstyle\sum_{i=1}^{n+1} I(i) \\ P^{(n+1)}_{\mu} &= \textstyle\sum_{i=1}^{n+1} P_\mu(i) \\ X^{(n+1)}_{\mu} &= \textstyle\sum_{i=1}^{n+1} X_\mu(i) \end{aligned} \tag{44}$$

This accounts for nine of the operators available in $\mathfrak{T}^{(n+1)}$. The remaining $9n$ operators must form n mutually orthogonal internal Lie algebras. It should be fairly evident that they are far from being unique. In fact, consider the example of $n = 2$. Then we have three basic particles which may be "coupled" 3! different ways:

$$\begin{array}{lll} (12)\,3, & (23)\,1, & (31)\,2, \\ (21)\,3, & (32)\,1, & (13)\,2. \end{array}$$

The couplings in the same column are equivalent in the sense that the internal generators of the two schemes differ only by minus signs. Let us consider the scheme (12) 3. Coupling particles 1 and 2, we obtain $\mathfrak{T}^{(2)}_{\text{ext}}$ and $\mathfrak{T}^{(2)}_{\text{int}}$ previously discussed which we may now denote by $\mathfrak{T}^{(3)}_{\text{ext}}(12)$ and $\mathfrak{T}^{(3)}_{\text{int}}(12)$, respectively, the superscript (3) showing that these Lie algebras are associated with a three-particle system. We now couple the system (12) with the particle 3 and obtain a second Lie algebra, $\mathfrak{T}^{(3)}_{\text{int}}(123)$, with the basis elements

$$\bar{I}(123) = I(12)\,I(3)\,[I(12) + I(3)]$$

$$\bar{P}_\mu(123) = P_\mu(12)\,I(3) - P_\mu(3)\,I(12)$$

and similarly for $\bar{X}_\mu(123)$; here $I(12) = I(1) + I(2)$, etc. It is immediately obvious that $\mathfrak{T}^{(3)}_{\text{int}}(12)$ and $\mathfrak{T}^{(3)}_{\text{int}}(123)$ are orthogonal. Thus we have constructed an external and two internal Lie algebras for our special example of three basic particles. It is clear that had we chosen any other coupling scheme enumerated above, say (23) 1, we would have obtained another pair of mutually orthogonal internal Lie algebras, namely, $\mathfrak{T}^{(3)}_{\text{int}}(23)$ and $\mathfrak{T}^{(3)}_{\text{int}}(231)$. Although the Lie algebras in each pair are orthogonal to each other, this is not true for two Lie algebras selected from each pair. Thus, e.g., $\mathfrak{T}^{(3)}_{\text{int}}(12)$ and $\mathfrak{T}^{(3)}_{\text{int}}(23)$ are *not* orthogonal.

Returning now to the general case, we define the total angular momentum of the $(n+1)$-particle system by

$$\begin{aligned} M^{(n+1)}_{\mu\nu} &= \textstyle\sum_{i=1}^{n+1} M_{\mu\nu}(i) \\ &= \textstyle\sum_{i=1}^{n+1} X_{[\mu}(i)\,P_{\nu]}(i)/I(i) \end{aligned} \tag{45}$$

as for the two-particle system treated above. Again, the total orbital angular momentum is

$$L^{(n+1)}_{\mu\nu} = X^{(n+1)}_{[\mu} P^{(n+1)}_{\nu]} / I^{(n+1)}$$

and the spin part of $M^{(n+1)}_{\mu\nu}$ is what is left after subtracting $L^{(n+1)}_{\mu\nu}$:

$$S^{(n+1)}_{\mu\nu} = M^{(n+1)}_{\mu\nu} - L^{(n+1)}_{\mu\nu}$$

Since both M and L are unique, so is S. We now show that S may be expressed (nonuniquely) entirely in terms of internal operators in the form

$$S^{(n+1)}_{\mu\nu} = \textstyle\sum_{i=1}^{n} S^{(n+1)}_{\mu\nu}(i)$$

$$S^{(n+1)}_{\mu\nu}(i) = \bar{X}_{[\mu}(i)\,\bar{P}_{\nu]}(i)/\bar{I}(i)$$

where $\bar{I}(i)$, $\bar{P}_\mu(i)$, $\bar{X}_\mu(i) \in \mathfrak{T}^{(n+1)}_{\text{int}}(i)$ for some choice of internal Lie algebras. The proof is by induction. We have already seen that the statement is true for $n = 1$. Suppose that it is true for n. Then

$$M^{(n)}_{\mu\nu} = L^{(n)}_{\mu\nu} + S^{(n)}_{\mu\nu}$$

and $S^{(n)}_{\mu\nu}$ is orthogonal to $\mathfrak{T}^{(n)}_{\text{ext}}$. Now

$$\begin{aligned} M^{(n+1)}_{\mu\nu} &= M^{(n)}_{\mu\nu} + M_{\mu\nu}(n+1) \\ &= L^{(n+1)}_{\mu\nu} + S^{(n)}_{\mu\nu} + S^{(n+1)}_{\mu\nu}(n) \end{aligned}$$

where

$$\begin{aligned} S^{(n+1)}_{\mu\nu}(n) = L^{(n)}_{\mu\nu} &- L^{(n+1)}_{\mu\nu} \\ &+ X_{[\mu}(n+1)\,P_{\nu]}(n+1)/I(n+1) \end{aligned} \tag{46}$$

But

$$L^{(n)}_{\mu\nu} = X^{(n)}_{[\mu} P^{(n)}_{\nu]} / I^{(n)}$$

$$L^{(n+1)}_{\mu\nu} = [X^{(n)}_{[\mu} + X_{[\mu}(n+1)]\,[P^{(n)}_{\nu]} + P_{\nu]}(n+1)]/[I^{(n)} + I(n+1)]$$

Substituting these expressions into Eq. (46) and simplifying, we find

$$S^{(n+1)}_{\mu\nu}(n) = \overline{X}^{(n+1)}_{\mu}(n)\,\overline{P}^{(n+1)}_{\nu}(n)/\overline{I}^{(n+1)}(n)$$

where

$$\overline{I}^{(n+1)}(n) = I^{(n)} I(n+1)\,[I^{(n)} + I(n+1)]$$

$$\overline{P}^{(n+1)}_{\mu}(n) = P^{(n)}_{\mu} I(n+1) - P_{\mu}(n+1)\, I^{(n)}$$

$$\overline{X}^{(n+1)}_{\mu}(n) = X^{(n)}_{\mu} I(n+1) - X_{\mu}(n+1)\, I^{(n)}$$

It is immediate that these generators are orthogonal to $\mathfrak{T}^{(n+1)}_{\text{ext}}$ spanned by $I^{(n+1)} = I^{(n)} + I(n+1)$, etc. Moreover, $S^{(n+1)}_{\mu\nu}(n)$ commutes with $S^{(n)}_{\mu\nu}$; this is because $S^{(n+1)}_{\mu\nu}(n)$ is a combination of operators from $\mathfrak{T}^{(n)}_{\text{ext}}$ and $\mathfrak{T}(n+1)$ with which $S^{(n)}_{\mu\nu}$ commutes, being by hypothesis a combination of internal operators with respect to $\mathfrak{T}^{(n)}_{\text{ext}}$. Thus the original statement is true for $n+1$ and the proof is complete.

Our next task is the construction of the internal (Lie) algebra $\mathfrak{E}^{(n+1)}_{\text{int}}$ for the $(n+1)$-particle system. By definition, it is the set of all operators in the enveloping algebra of $\mathfrak{T}^{(n+1)}$ which commute with all of $\mathfrak{P}^{(n+1)}_{\text{ext}}$ spanned by Eq. (44) and (45). Let us suppose that we have made a definite choice of internal Lie algebras $\mathfrak{T}^{(n+1)}_{\text{int}}(1), \cdots, \mathfrak{T}^{(n+1)}_{\text{int}}(n)$ for our system. Dropping the superscript $(n+1)$ on the understanding that n is fixed until further notice, we define

$$A^{\pm}_{\mu}(i) = [\overline{P}_{\mu}(i) \pm i\overline{X}_{\mu}(i)]\,[2\overline{I}(i)]^{-1/2} \quad i = 1, 2, \cdots, n$$

in analogy with Eq. (42). We have the commutation relations

$$[A^{+}_{\mu}(i), A^{-}_{\nu}(j)] = g_{\mu\nu}\,\delta_{ij}$$

$$[A^{\pm}_{\mu}(i), A^{\pm}_{\nu}(j)] = 0$$

We may express the spin operators in terms of the A's:

$$S_{\mu\nu}(i) = -iA^{+}_{[\mu}(i)\,A^{-}_{\nu]}(i) \tag{47}$$

$$S_{\mu\nu} = \textstyle\sum^{n}_{i=1} S_{\mu\nu}(i) \tag{48}$$

It follows from above that

$$[S_{\mu\nu}(i), S_{\rho\sigma}(j)] = i\,\delta_{ij}\, S_{[\mu[\sigma}(i)\, g_{\rho]\nu]}$$

The basis of the enveloping algebra $\mathfrak{E}_{\text{ext}}$ of $\mathfrak{P}_{\text{ext}}$ consists of the operators I, P_{μ}, X_{μ}, and $M_{\mu\nu}$ or, alternately, of I, P_{μ}, X_{μ}, and $S_{\mu\nu}$. That is to say, every operator in $\mathfrak{E}_{\text{ext}}$ is a polynomial (or a formal limit of such polynomials) in operators of either basis set. It is clear that the operators $A^{\pm}_{\mu}(i)$ commute with I, P_{μ}, and X_{μ} but not with $S_{\mu\nu}$. In fact, we have

$$[S_{\mu\nu}, A^{\pm}_{\rho}(i)] = iA^{\pm}_{[\mu}(i)\, g_{\nu]\rho}$$

as expected. Thus the only internal operators of the $(n+1)$-particle system with respect to the group P are the various I's, the Casimir operators of P, and the Lorentz scalars or invariants constructed from the A-operators. We consider the last-mentioned set of internal operators. Just as before, we define

$$\xi^{ij} = A^{+}(i) \cdot A^{+}(j)$$

$$\xi_{ij} = A^{-}(i) \cdot A^{-}(j)$$

$$\xi^{i}_{\ j} = A^{+}(i) \circ A^{-}(j)$$

It is trivial to verify that

$$[\xi^{ij}, \xi^{kl}] = [\xi_{ij}, \xi_{kl}] = 0$$

$$[\xi^{ij}, \xi^{k}_{\ l}] = \delta^{(i}_{\ l}\xi^{j)k}$$

$$[\xi^{ij}, \xi_{kl}] = \delta^{(i}_{(k}\xi^{j)}_{\ l)}$$

$$[\xi^{i}_{\ j}, \xi^{k}_{\ l}] = \delta^{i}_{\ l}\xi^{k}_{\ j} - \delta^{k}_{\ j}\xi^{i}_{\ l}$$

$$[\xi^{i}_{\ j}, \xi_{kl}] = \delta^{i}_{\ (k}\xi_{l)j} \tag{49}$$

where parentheses denote symmetrizations:

$$a_{(i}b_{j)} = a_i b_j + a_j b_i$$

Moreover, we have

$$\xi^{ij} = \xi^{ji} = (\xi_{ij})^{*}$$

$$\xi^{i}_{\ j} = (\xi^{j}_{\ i})^{*}$$

In addition to the ξ's there exist further invariants constructed with the help of the antisymmetric tensor $\varepsilon^{\mu\nu\rho\sigma}$:

$$\eta^{ijkl} = \varepsilon^{\mu\nu\rho\sigma} A^+_\mu(i) A^+_\nu(j) A^+_\rho(k) A^+_\sigma(l)$$

$$\equiv [A^+(i)\, A^+(j)\, A^+(k)\, A^+(l)]$$

We adopt the convention that an index on η^{ijkl} is lowered whenever the operator A^+ associated with that particular index is replaced by A^-. Thus, e.g.,

$$\eta^i{}_j{}^{kl} = [A^+(i)\, A^-(j)\, A^+(k)\, A^+(l)]$$

It should be noted that the various A-operators may freely be commuted within the brackets defining η's since the commutator of two A's is either zero or involves the symmetric metric tensor $g_{\mu\nu}$; the cost of interchanging two adjacent A's is a minus sign. Hence we may always write η's in one of the following canonical forms:

$$\eta^{ijkl},\ \eta^{ijk}{}_l,\ \eta^{ij}{}_{kl},\ \eta^i{}_{jkl},\ \eta_{ijkl}$$

In each case the η's are completely antisymmetric in both upper and lower indices separately. Taking the hermitian conjugate of a given η amounts to lowering upper and raising lower indices, besides interchanging the order of all indices; e.g.,

$$(\eta^{ij}{}_{kl})^* = \eta^{lk}{}_{ji}$$

The commutation relations of the η's are rather complicated and will not be needed in this report.

It may be worthwhile to point out that the Casimir operators of $\mathfrak{P}_{\text{ext}}$ may be expressed in terms of the ξ's and η's as follows. From Eq. (33), (34), (47), and (48) we find

$$F = \frac{1}{2} \sum_{i,j=1}^{n} A^+_{[\mu}(i) A^-_{\nu]}(i)\, A^{+[\mu}(j)\, A^{-\nu]}(j)$$

$$= \frac{1}{2}(\xi^{ij}\xi_{ij} + \xi_{ij}\xi^{ij}) - \xi^i_j\xi^j_i - 2n(n-1)$$

$$G = -\frac{1}{4} \sum_{i,j=1}^{n} \varepsilon^{\mu\nu\rho\sigma} A^+_{[\mu}(i) A^-_{\nu]}(i) A^+_{[\rho}(j) A^-_{\sigma]}(j)$$

$$= \eta^{ij}{}_{ij}$$

where summations over repeated indices are understood to run from 1 to n. For the special case $n = 1$ we have

$$F = \frac{1}{2}(\xi^{11}\xi_{11} + \xi_{11}\xi^{11}) - (\xi^1_1)^2$$

$$= 4(K_1^2 + K_2^2 - K_3^2)$$

$$= 4Q$$

as stated without proof previously.

Disregarding the I's, we see that the internal enveloping algebra $\mathfrak{E}_{\text{int}}$ is the formal closure of all polynomials in the ξ's and η's. It should be clear that $\mathfrak{E}_{\text{int}}$ is an infinite-dimensional Lie algebra and as such is not very useful. What we need is a *finite*-dimensional Lie algebra of internal symmetries, an analog of $\mathfrak{L}^{(3)}_0$ in the two-particle case discussed above. Such algebra, $\mathfrak{P}_{\text{int}}$, is generated by all ξ^{ij}, ξ^i_j, and ξ_{ij}, $i, j = 1, 2, \cdots, n$; the number of ξ's is easily seen to be $(2n + 1)\,n$. From the commutation relations (Eq. 49) one finds that $\mathfrak{P}_{\text{int}}$ is isomorphic to the Lie algebra $\boldsymbol{sp}(n)$ of the symplectic group $Sp(n)$ (Ref. 50) of $2n \times 2n$ complex unitary matrices M obeying

$$MJ_nM = J_n$$

here M is the transpose of M and

$$J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

I_n being the $n \times n$ unit matrix. Since $Sp(1)$ is just $L^{(3)}_0$ and is contained in every $Sp(n)$, $n \geqq 1$, as a non-compact subgroup, it follows that each $Sp(n)$ is also not compact.[27] The non-compactness of internal symmetry groups we have obtained is to be traced back to the Lorentz metric ($g_{\mu\nu}$) associated with space-time.

The set of all ξ^i_j (n^2 in number) generates a ***maximal compact subalgebra*** of $\boldsymbol{sp}(n)$; it is just the Lie algebra of the unitary group $U(n) \simeq U(1) \times SU(n)$. All unitary representations of $U(n)$ are finite-dimensional, labeled by discrete quantum numbers, and are adequately discussed in the literature. Before proceeding with an analysis of internal symmetries just obtained, we wish to discuss how they could be interpreted physically.

We envisage a situation in which physical one-particle states are described mathematically more and more accurately by increasing the number of basic state vectors

[27]As is well known, the compactness or noncompactness of a group depends not only on the commutation relations of its generators but also on their behavior under hermitian conjugation.

$|\sigma p\rangle$ from which the state vectors of physical particles are constructed. Thus we would have the hierarchy of state vectors (or wave functions) $\phi^{(0)}, \phi^{(1)}, \phi^{(2)}, \cdots$, all describing the same given particle but with progressively greater detail, i.e., by means of a progressively larger number of internal quantum numbers. This hierarchy of state vectors is associated with the hierarchy of internal groups:

$$\begin{array}{ccccccc} Sp(1) & \subset & Sp(2) & \subset & Sp(3) & \subset & \cdots \\ \cup & & \cup & & \cup & & \\ U(1) & \subset & U(2) & \subset & U(3) & \subset & \cdots \end{array}$$

Within any fixed-n approximation, the quantum numbers associated with $U(n)$ and its subgroups may be used to label the state vectors. Since $U(n)$ is not the full symmetry group in this approximation, we expect that $U(n)$-symmetry breaking should be caused by those generators of the super-group $Sp(n)$ of $U(n)$ which are not in $U(n)$.[28] Just as in the previously discussed case of $Sp(1)$, we shall argue that continuous quantum numbers stemming from the non-compactness of $Sp(n)$ will have to be integrated out in forming "wave packets" in order that one obtain a normalizable internal wave function or state vector. In other words, for *physical particles*, only discrete quantum numbers associated with $U(n)$ are available to label state vectors. We now see in principle how a "hidden" symmetry-breaking mechanism could operate in the realm of physical particles.

We wish now to investigate a possible scheme of labeling internal parts of one-particle state vectors. As we have seen, from $n+1$ basic Lie algebras $\mathfrak{T}(i)$ we can construct one T-external and n T-internal Lie algebras with four-momentum operators P_μ and $P_\mu(1), \cdots, P_\mu(n)$. The total number of components of these four-vector operators is $4(n+1)$, precisely the number of diagonal operators in $\mathfrak{P}_{\text{int}}$ and $\mathfrak{P}_{\text{ext}}$ (omitting the σ's). Disregarding P_μ, S^2, and S_3, we have to exhibit $N = 4n - 2$ commuting internal operators for each $n \geqq 1$.

The case of $n = 1$ of two basic particles has already been adequately discussed. For $n = 2$ we have to construct $N = 6$ operators. Now

$$S_{\mu\nu} = S_{\mu\nu}(1) + S_{\mu\nu}(2)$$

$$S_{\mu\nu}(i) = -iA^{+}_{[\mu}(i)A^{-}_{\nu]}(i)$$

[28] Such breaking should of course be compatible with known exact conservation laws.

It is easily seen that the operators

$$F = -\frac{1}{2}S_{\mu\nu}S^{\mu\nu} = F(1) + F(2) - S_{\mu\nu}(1)\,S^{\mu\nu}(2)$$

$$F(1) = -\frac{1}{2}S_{\mu\nu}(1)\,S^{\mu\nu}(1)$$

$$F(2) = -\frac{1}{2}S_{\mu\nu}(2)\,S^{\mu\nu}(2)$$

$$G = \frac{1}{2}S_{\mu\nu}\tilde{S}^{\mu\nu} \equiv S_{\mu\nu}(1)\,\tilde{S}^{\mu\nu}(2)$$

are simultaneously diagonalizable. Let

$$I_1 = -\frac{1}{2}(\xi^1_2 + \xi^2_1)$$

$$I_2 = \frac{i}{2}(\xi^2_1 - \xi^1_2)$$

$$I_3 = \frac{1}{2}(\xi^1_1 - \xi^2_2)$$

$$B = \xi^1_1 + \xi^2_2 \tag{50}$$

One verifies that

$$[B, I_k] = 0, \qquad k = 1, 2, 3$$

$$[I_i, I_j] = ie_{ijk}I_k$$

Thus one may identify B with the baryon (or, for that matter, lepton) number and the I_i with the three components of isospin. As we shall later explain, this identification is not quite unique; we disregard this point for the moment. The operators B and I_i generate

$$U(1) \times SU(2) \simeq U(2)$$

It is well known that B, I_3, and

$$\mathbf{I}^2 = \frac{1}{2}\xi^i_j\xi^j_i - \frac{1}{4}(\xi^i_i)^2$$

$$= \frac{1}{2}\xi^i_j\xi^j_i - \frac{1}{4}B^2$$

form a maximal commuting set of operators for $U(2)$. The question now arises whether the sets $\{F, G, F(1), F(2)\}$ and $\{B, I_3, \mathbf{I}^2\}$ commute, i.e., whether each operator of one set commutes with every operator of the other set.

With a little algebra we see that this is not the case. Namely,

$$[F(i), \mathbf{I}^2] \neq 0 \qquad \text{for } i = 1, 2$$

This is of course fortunate for otherwise we would have had seven commuting operators instead of the expected maximum of six. If we insist on diagonalizing $\mathbf{I}^2$, then we must find an extra commuting operator to augment the set $\{F, G, B, I_3, \mathbf{I}^2\}$. Let

$$A = \xi^{ij}\xi_{jk}\xi^k_i$$

Then, as is easily verified,

$$A^* = A$$

$$[A, \xi^i_j] = 0, \qquad i, j = 1, 2$$

Since B, I_3, and $\mathbf{I}^2$ are linear or bilinear in the ξ^i_j, it follows that A commutes with the former operators and, of course, with F and G, the Casimir operators of $Sp(2)$. We are now in a position to introduce eigenvectors of our set of eight commuting operators:

$$(F, G, A, B, \mathbf{I}^2, I_3, \mathbf{S}^2, S_3)\,|\,k\nu a b I \iota s \mu\rangle$$
$$= (1 + \nu^2 - k^2, 2k\nu, a, b, I(I+1), \iota, s(s+1), \mu) \cdot |\,k\nu a b I \iota s \mu\rangle$$

here k, ν, s, and μ have the values discussed in Section IV and

$$b = 0, \pm 1, \pm 2, \cdots$$

$$I = 0, \tfrac{1}{2}, 1, \cdots$$

$$\iota = -I, -I+1, \cdots, I$$

The eigenvalues a of A are at least partly continuous [from general arguments based on the non-compactness of $Sp(2)$]; the precise spectrum of A does not concern us here. Physically meaningful are the discretely normalizable internal state vectors of the form

$$|\,b I \iota s \mu\rangle = \sum_k \int d\nu \int da\, w(k\nu a)\,|\,k\nu a b I \iota s \mu\rangle$$

just as in the previously discussed case of compositions of two basic particles.

We note that the identification of $\tfrac{1}{2}(\xi^1_1 - \xi^2_2)$ as the third component of physical isospin is arbitrary to within the following unitary transformations of the ξ's:

$$\xi^{ij} \to U^{-1}\xi^{ij}U$$

$$\xi^i_j \to U^{-1}\xi^i_jU$$

$$\xi_{ij} \to U^{-1}\xi_{ij}U \tag{51}$$

where

$$U = \exp[-i\theta(A)]$$

for some $A \in \mathbf{sp}(2)$. The "direction" of isospin in the group space of $Sp(2)$ is thus completely undetermined by our essentially "kinematical" considerations. How, then, is this direction to be fixed? We believe that a full answer to this question can be given only in the framework of a dynamical theory. The following comment might, however, be appropriate. We know that S-matrix elements have the form $\langle \text{in}\,|\,S\,|\,\text{in}\rangle = \langle \text{out}\,|\,\text{in}\rangle$. It is clear that one can choose the direction of I_3, e.g., arbitrarily for one set of state vectors, say, for the incoming ones. The simplest such choice would of course be that given by Eq. (50). The direction of the third component of isospin for outgoing vectors would in general be different from that for incoming vectors; it would be determined by the S-matrix dynamics or, in our theory, simply by an internal rotation in the space of an appropriate internal group through some angle consistent with crossing principle and/or some additional constraints.

From Eq. (50) we find a formula for the electric charge number:

$$Q/e = I_3 + \frac{1}{2}B = \xi^1_1 = 0, \pm 1, \pm 2, \cdots \tag{52}$$

in our three-basic particle approximation ($n = 2$) of physical state vectors. What is the interpretation of ξ^1_1 in the $n = 1$ approximation? One might naively expect that ξ^1_1 is still Q/e. Note, however, that now $I_i = 0$ and hence $Q/e'' = ''B$ from the above formula. This of course is nonsense and simply means that we cannot infer the physical significance of ξ^1_1 for $n = 1$ from that for $n = 2$. Rather, we may argue as follows. Strong interactions dominate electromagnetic ones in strength. Thus we may expect that the baryon number should manifest itself before the electric charge number in any scheme of approximation of physical state vectors. On these grounds we identify $\xi^1_1 = 2\kappa$ with B for $n = 1$. In general, we shall have the following identifications:

$$U(1): B = \xi^1_1$$

$$U(2): B = \xi^1_1 + \xi^2_2$$

$$U(3): B = \xi^1_1 + \xi^2_2 + \xi^3_3 \qquad (53)$$

Let us quickly examine the case $n = 3$. The relevant internal groups are $Sp(3)$ and $U(3)$ with a total of $N = 4 \times 3 - 2 = 10$ commuting operators:

$$B = \xi^1_1 + \xi^2_2 + \xi^3_3$$

$$I_3 = \frac{1}{2}(\xi^1_1 - \xi^2_2)$$

$$Y = \xi^3_3$$

$$\mathbf{I}^2 = \frac{1}{4}[(\xi^1_1)^2 + 2\xi^1_2\xi^2_1 + 2\xi^2_1\xi^1_2 + (\xi^2_2)^2]$$

$$C_1 = \xi^i_j \xi^j{}_i$$

$$C_2 = \xi^i_j \xi^j_k \xi^k_i$$

$$A_1 = \xi^{ij}\xi_{jk}\xi^k_i$$

$$A_2 = \xi^{ij}\xi_{jk}\xi^k_l \xi^l_i$$

$$F = -\frac{1}{2}S_{\mu\nu}S^{\mu\nu}$$

$$G = \frac{1}{2}S_{\mu\nu}\tilde{S}^{\mu\nu}$$

Now $S_{\mu\nu}$ consists of three parts and the sums over repeated Latin indices run over 1, 2, 3. C_1 is just the total "F-spin" squared in the terminology of Gell-Mann (Ref. 51), while C_2 is the second Casimir operator of $SU(3)$. To form normalizable wave packets one now integrates or sums over the eigenvalues of A_1, A_2, F, and G. Again, there are non-uniqueness problems in identifying the *physical* isospin generators, much as in the previously discussed case $n = 2$.

It should now be fairly clear how to handle the case of an arbitrary number of basic particles. We shall not pursue this matter any further. Instead, let us briefly discuss how discrete quantum numbers, such as parity, are to be treated in our theory. Consider the $n = 2$ approximation of a physical particle. The parity operation σ, according to Eq. (12), has the following effect on *each* of the three basic particles:

$$\sigma: \quad \begin{array}{l} P_\mu(i) \to P^\mu(i) \\ X_\mu(i) \to X^\mu(i) \qquad (i = 1, 2, 3) \\ I(i) \to I(i) \end{array}$$

But this means that external as well as internal operators transform non-trivially under the parity operation:

$$\sigma: \quad \begin{array}{l} P_\mu \to P^\mu \\ P_\mu(1) \to P^\mu(1) \\ P_\mu(2) \to P^\mu(2) \end{array}$$

and similarly for X's. Now

$$|ps\mu;k\nu abI_\iota\rangle = \int d\bar{p}_1 \int d\bar{p}_2 \,|p;\bar{p}_1\bar{p}_2\rangle\langle\bar{p}_1\bar{p}_2|s\mu k\nu abI_\iota\rangle \qquad (54)$$

Let J_σ be the parity operator acting on $\mathfrak{H}^\infty$. Applying it to the above state vector, letting $\bar{\mathbf{p}}_1 \to -\bar{\mathbf{p}}_1$, $\bar{\mathbf{p}}_2 \to -\bar{\mathbf{p}}_2$ in the integrand, and noting that $\int d\bar{p}$ is invariant under $\bar{\mathbf{p}} \to -\bar{\mathbf{p}}$, we find

$$J_\sigma|ps\mu;k\nu abI_\iota\rangle = \int d\bar{p}_1 \int d\bar{p}_2 \,|{}^\sigma p;\bar{p}_1\bar{p}_2\rangle\langle{}^\sigma\bar{p}_1\,{}^\sigma\bar{p}_2|s\mu k\nu abI_\iota\rangle$$

The transformation coefficient under the integrand satisfies a set of differential equations in the eight variables $\bar{p}_1$ and $\bar{p}_2$ and as such it will have certain symmetry properties with respect to the transformation $\bar{\mathbf{p}}_{1,2} \to -\bar{\mathbf{p}}_{1,2}$. This is analogous to the well known transformation property $\langle\mathbf{n}|s\mu\rangle = Y_{s\mu}(\mathbf{n}) \to (-)^s Y_{s\mu}(\mathbf{n})$ under $\mathbf{n} \to -\mathbf{n}$, where $\mathbf{n}$ is a unit vector. The precise behavior of $\langle\bar{p}_1\bar{p}_2|s\mu k\nu abI_\iota\rangle$ under $\bar{\mathbf{p}}_{1,2} \to -\bar{\mathbf{p}}_{1,2}$ does not concern us at the moment. The important point is that it will transform into itself times a phase which may possibly depend not only on s but also on the other quantum numbers. This phase η, whatever it will turn out to be, is to be interpreted as the intrinsic (or internal) parity of a particle represented by the state vector (Eq. 54).[29] It is not unreasonable to guess that η will depend on b and I in addition to s. We defer further consideration of this question to future work on analytical aspects of our theory.

[29]More precisely, by a wave packet of such state vectors.

How would one obtain parity-violating Poincaré-invariant interactions in our theory? The answer is easy to see. We merely note that the operators η^{ijkl} are Lorentz pseudoscalars:

$$\begin{aligned}\eta^{ijkl} &= \varepsilon^{\mu\nu\rho\sigma} A^{+}_{\mu}(i) A^{+}_{\nu}(j) A^{+}_{\rho}(k) A^{+}_{\sigma}(l) \\ &\xrightarrow{\sigma} \varepsilon^{\mu\nu\rho\sigma} A^{+\mu}(i) A^{+\nu}(j) A^{+\rho}(k) A^{+\sigma}(l) \\ &= -\eta^{ijkl}\end{aligned}$$

since $\varepsilon^{\mu\nu\rho\sigma} = -\varepsilon_{\mu\nu\rho\sigma}$. Parity violation[30] is obtained if the Hamiltonian or the S-operator contains terms involving odd powers of η's. We need at least two T-internal Lie algebras in order to construct non-vanishing η's. Since strong interactions are governed by a single T-internal Lie algebra (Ref. 52), it follows that no parity violation is possible for them.

[30]Note that also $\eta \xrightarrow{T} -\eta$.

Under the anti-automorphism γ, each $P_\mu(i) \to -P_\mu(i)$, $X_\mu(i) \to -X_\mu(i)$, and $I(i) \to -I(i)$. Thus $S_{\mu\nu} \to -S_{\mu\nu}$, $\overline{P}_\mu(i) \to \overline{P}_\mu(i)$, $\overline{X}_\mu(i) \to \overline{X}_\mu(i)$, $\overline{I}(i) \to -\overline{I}(i)$ and hence $\xi^i_i = [\overline{P}(i)^2 + \overline{X}(i)^2]/2\overline{I}(i) \to -\xi^i_i$. This means that the operators S_3, Q, B, I_3, Y acquire a minus sign under γ. It is therefore quite consistent to regard γ as a particle-antiparticle conjugation operator in our formalism.

VII. DISCUSSION

In summarizing the work and results of preceding sections, we shall adopt here a different attitude toward our theory. Namely, we shall take the basic group T as the point of departure without reviewing the reasons which led us to this group; they are adequately discussed, we believe, in Sections II and IV.

It may be appropriate to offer a few comments regarding the nature of T itself. Let us introduce the column vector $\xi = \mathrm{col}(\phi(x), x, 1)$, where ϕ is some real-valued function of the four-vector x specifying the time and position of an event in space-time. Applying to ξ the matrix (Eq. 36) corresponding to the element (α, v, a, l) of the augmented Poincaré group P, we find

$$\begin{aligned}x \to x' &= lx + a \\ \phi(x) \to \phi(x') &= \phi(x) + v \cdot lx + \alpha\end{aligned}$$

This shows that the action of P on the Lorentz space L is just that of the Poincaré group P_1. Of course, T transforms L in the same way as does the translation subgroup T_0 of P_1. Thus the customary geometry of flat space-time has not been tampered with in going over from P_0 to P and T, and this is most gratifying. Yet, something new has been added, the function $\phi(x)$ associated with each point in space-time. At present we do not understand its physical significance.

In Section VI we have shown that the representation theory of T yields in a relatively straightforward manner the hierarchy of noncompact internal groups $Sp(1) \subset Sp(2) \subset \cdots \subset Sp(n) \subset \cdots$. We have presented arguments that only the maximal compact subgroups $U(n)$ of each $Sp(n)$ are of significance in providing internal symmetries for *physical* particles. The hierarchy $SU(1) \subset SU(2) \subset \cdots$, related to $U(1) \subset U(2) \subset \cdots$ in an obvious way, has been considered on empirical grounds by Neville (Ref. 53). The relevance of unitary groups of low n to particle physics is now quite well established. It is true that these groups fail to provide exact symmetries, because they are more or less badly broken in nature. Nevertheless, they furnish very useful approximate classification schemes of particles. It will be interesting to see whether a dynamical theory can be constructed which will allow one to understand the detailed mechanism of symmetry breaking.[31] What we have in mind is a dynamics based purely on the group-theoretical methods employed in this work. To see intuitively the feasibility of such approach, we must examine the role played in our theory by states of spacelike momenta.

If p is timelike or lightlike, then the state vector $|p;\alpha\rangle$ may be thought to represent a matter wave of mass $m = (p^2)^{1/2}$ and momentum $\mathbf{p}$, with all other quantum numbers indicated by α. On the other hand, if p is spacelike then we have no physical intuition to guide us except

[31]For discussions of $SU(3)$ symmetry breaking see Ref. 54, and also Ref. 55–56.

the notion that such momenta are somehow associated with virtual particles and interactions. To make the picture clearer, let us consider the elastic scattering of two nucleons through the exchange of a single pion. A diagrammatic representation of this process is given in Fig. 1. Here V denotes a vertex operator containing form factors, gamma matrices, etc. The amplitude for the process is proportional to $(p_5^2 - m_\pi^2)^{-1}$, where $p_5^2 = (p_1 - p_3)^2 < 0$; i.e., the exchanged "pion" carries spacelike momentum. To see how spacelike pions would manifest themselves in our formalism, let us first distort the diagram of Fig. 1 into that of Fig. 2. Let $|a\rangle$, $|b\rangle$, and $|c\rangle$ be the states of our scattering system corresponding to the dashed lines in Fig. 2. At a the two initial nucleons are both free, and their combined state is represented by $|a\rangle = |p_1\alpha_1, p_2\alpha_2\rangle$. Subsequently, a pion is emitted or absorbed by nucleon 1,

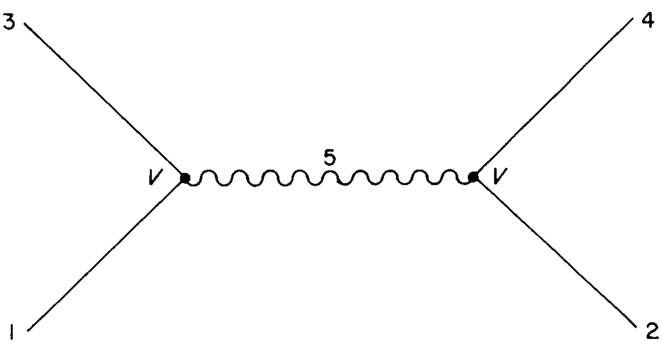

Fig. 1. Nucleon-nucleon scattering through an exchange of a virtual pion

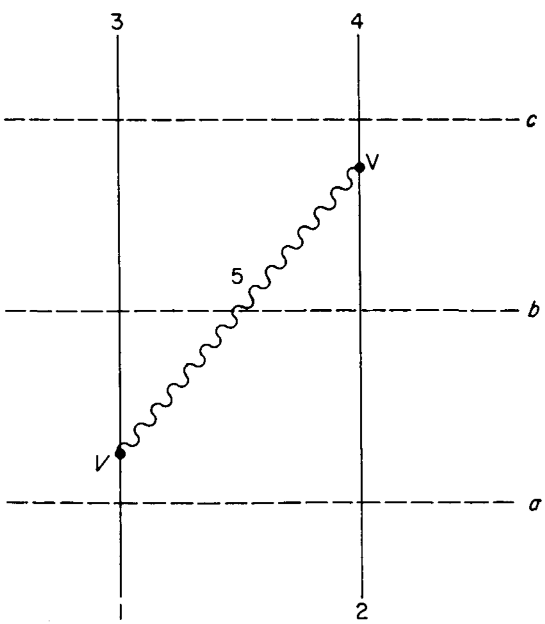

Fig. 2. A redrawing of the diagram of Fig. 1

and at b we find a state of two free nucleons and one virtual pion; thus $|b\rangle = |p_3\alpha_3, p_5\alpha_5, p_2\alpha_2\rangle$. Finally, both nucleons are free and no virtual pions are present: $|c\rangle = |p_3\alpha_3, p_4\alpha_4\rangle$. Note that the diagram of Fig. 2 does *not* indicate whether the pion is first emitted by nucleon 1 and then absorbed by 3 or emitted by 3 and subsequently absorbed by 1. That is to say, this diagram gives no information about the temporal evolution of the system during the interaction. The state vectors $|a\rangle$ and $|c\rangle$ represent free stable particles and are perfectly legitimate in the strict on-the-mass-shell S-matrix theory of strong interactions. Vectors of the form $|b\rangle$, on the other hand, are not admitted in this theory, since they contain virtual pions for which $p_5^2 = m_\pi^2$ is not satisfied. That is to say, matrix elements of the form $\langle b|S|a\rangle$ are taboo. In the S-matrix theory all masses are kept at fixed physical values, and only various invariant energy and momentum transfer variables are allowed to vary. In reality, continuation in external masses frequently has to be resorted to, e.g., when dealing with anomalous thresholds.

We envisage a different kind of "S-matrix theory" based on our group-theoretical formalism. Namely, external as well as internal masses are allowed full freedom of variation without any *a priori* constraints (except for an overall energy-momentum conservation). The problem now becomes to show, if possible, that only certain special values of external masses are consistent with the group structure of the theory. Any continuation in either external or internal masses is to be made by means of operators of the form $\exp(-iv \cdot X)$. The fact that these operators have an effect not only on masses but also on various other quantum numbers indicates that one may expect very intricate dynamical correlations between external and internal degrees of freedom of particles.

Is there any way we can understand the physical significance of the identity operator I of the basic Lie algebra $\mathfrak{X}$? As noted in Section IV, $\mathfrak{X}$ is a covariant generalization of the canonical commutation relations of quantum mechanics:

$$[P_i, P_j] = [X_i, X_j] = 0$$

$$[P_i, X_j] = -i\delta_{ij} \rightarrow [P_\mu, X_\nu] = ig_{\mu\nu}I$$

The appearance of I is inescapable if we are to play the game of Lie algebras. Physical particle state vectors are assumed to be eigenvectors of I with unit eigenvalue: $\sigma = 1$. With this choice of σ the relation between momentum and configuration-space wave functions is through Fourier transforms involving $\exp(\pm ip \cdot x/\hbar)$, with $\hbar$ having the conventional value of 1.054×10^{-27} erg sec. It is clear that we could have chosen σ to be of any finite positive value; instead of $\hbar$ we would have then been obliged to use $\hbar' = \hbar/\sigma$. If now a physical particle is approximated by a composition of $n + 1$ basic particles, then we must have

$$\sigma = \sum_{i=1}^{n+1} \sigma_i = 1$$

where all $\sigma_i > 0$ in accordance with the arguments of Section IV. Thus all $\sigma_i < 1$. This fact has some very interesting consequences for the commutators σ of T-internal Lie algebras. Suppose a physical particle, in the lowest order approximation, is composed of two basic particles. Then $\sigma_1 + \sigma_2 = 1$ and hence

$$\sigma = \sigma_1\sigma_2 (\sigma_1 + \sigma_2) \leqq 1/4$$

This means that the internal Lie algebra $\mathfrak{T}^{(2)}_{\text{int}}$ is at least four times more "weakly quantized" than the external algebra $\mathfrak{T}^{(2)}_{\text{ext}}$. Suppose we now take four basic particles and use the coupling scheme

$$\begin{tikzcd}[row sep=small, column sep=small]
1 \arrow[dr] & & \\
& 5 \arrow[ddr] & \\
2 \arrow[ur] & & \\
& & 7 \\
3 \arrow[dr] & & \\
& 6 \arrow[uur] & \\
4 \arrow[ur] & &
\end{tikzcd}$$

Then, on the average, $\sigma_i \simeq 1/4$ and so

$$\sigma_5 = \sigma_1 + \sigma_2 \simeq 2^{-1}$$
$$\bar{\sigma}_5 = \sigma_1\sigma_2 (\sigma_1 + \sigma_2) \simeq 2^{-5}$$
$$\sigma_6 = \sigma_3 + \sigma_4 \simeq 2^{-1}$$
$$\bar{\sigma}_6 = \sigma_3\sigma_4 (\sigma_3 + \sigma_4) \simeq 2^{-5}$$

Coupling 5 and 6, we get

$$\sigma_7 = \sigma_5 + \sigma_6 = 1$$
$$\bar{\sigma}_7 = \sigma_5\sigma_6 (\sigma_5 + \sigma_6) \simeq 2^{-2}$$
$$\bar{\sigma}_7' = \bar{\sigma}_5 + \bar{\sigma}_6 \simeq 2^{-4}$$
$$\bar{\sigma}_7'' = \bar{\sigma}_5\bar{\sigma}_6 (\bar{\sigma}_5 + \bar{\sigma}_6) \simeq 2^{-14}$$

Thus we find a rather striking hierarchy of T-internal Lie algebras with progressively more and more "classical" commutators.[32] For larger numbers of basic particles, the hierarchy is of course even more striking. Thus, e.g., for eight basic particles, we reach $\sigma \simeq 2^{-68} \simeq 3.4 \times 10^{-21}$. What is the significance of these weak commutators? Can we expect these extremely small values of $\bar{\sigma}$'s to manifest themselves in physically interpretable numerical answers? We don't know yet. However, for what they are worth, we offer the following speculations bearing on these questions.

Suppose a physical particle is composed of a very large number of basic particles. Then it is intuitively reasonable to argue that on the average a single basic particle contributes very little to the internal structure of the composite particle. In particular, its coupling to the remainder of the composite particle is expected to be quite weak in the sense that it should not make much difference whether one approximates the composite particle by 100 basic particles or 99, say. If we are content to describe only the gross features of internal structure of the physical particle, then, as a first approximation, we would presumably "split" it into two roughly equal parts and investigate structural effects due to their relative motion. Interactions between the two parts should be called strong, if anything. Now we could subdivide each of the two parts and thus get more structure due to additional internal modes obtained. If we allow ourselves the luxury of classical pictures, we may imagine the situation as shown in Fig. 3. Particles 1 and 2 are coupled to form the subsystem (12), and similarly for 3 and 4. There are three different internal motions shown in this picture, namely, the internal motions of subsystems (12) and (34) and the relative motion of these subsystems with respect to each other. Intuitively, we would expect that the coupling between internal modes of (12) and (34)

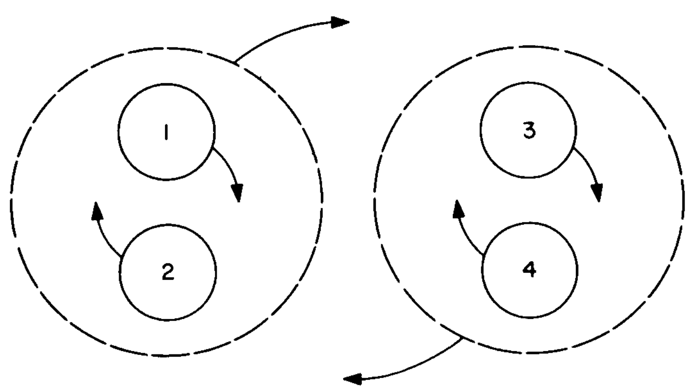

Fig. 3. Classical picture of internal motion of a composite particle approximated by four basic particles

[32]The particular coupling scheme used is irrelevant. Similar conclusions obtain in the case of any coupling scheme although one need not have the same set of values of σ's.

("second-order internal to second-order internal coupling") should be weaker than, e.g., between those of (12) and [(12)(34)] ("second-order internal to first-order internal coupling"), whatever be the nature of these couplings. We might further suspect that the strength of a particular coupling should be related to the particular pair of Lie algebras describing it. To see how this could happen we investigate the interaction of two physical particles approximated, for the sake of simplicity, by a basic particle each. It is possible to show (Ref. 52) that the T-matrix elements $\langle 34|T|12\rangle \equiv T$ for the reaction $1 + 2 \rightarrow 3 + 4$ are functions of $\lambda = \bar{p}^2/\bar{\sigma}$, among other things. Now

$$\lambda = (p_1\sigma_2 - p_2\sigma_1)^2/\bar{\sigma} = [(m_1^2\sigma_2 + m_2^2\sigma_1)\,\sigma - \sigma_1\sigma_2 s]/\bar{\sigma}$$

where $s = (p_1 + p_2)^2$. Suppose the reaction in question can proceed via an intermediate particle: $1 + 2 \rightarrow 5^* \rightarrow 3 + 4$. Then the coupling constant of this particle to channels (12) and (34) (assuming strictly elastic scattering for simplicity) is given by (Ref. 57)

$$\frac{1}{g^2} = \frac{d}{ds}\,\Re e\,\frac{1}{T(\lambda)}\bigg|_{s = m_5^2}$$

$$= \frac{1}{\sigma} f(\lambda_5)$$

where

$$f(\lambda) = -\frac{d}{d\lambda}\,\Re e\,\frac{1}{T(\lambda)}$$

Thus g^2 is proportional to $\sigma = 2$. Since the only way two basic particles can interact is strongly (because they have no electric charges, etc.), it follows that $\sigma \sim g^2$ characterizes the strength of nuclear or strong interactions. The function $T(\lambda)$ depends on various σ's only implicitly through $\lambda = \lambda(\sigma_1, \sigma_2, m_1, m_2, s)$ and hence so does $f(\lambda)$. If $f(\lambda)$ is a reasonably slowly varying function of λ in some neighborhood of values of m_1, m_2, and s (for fixed σ_1 and σ_2), corresponding to physical hadron masses (we are excluding particles with atomic numbers $A > 1$), then the various hadronic coupling constants are of the same order of magnitude. This is of course the case experimentally. We may think of σ as setting the *scale* of physical coupling constants for strong interactions; the function $f(\lambda)$ then accounts for *variations* of coupling strength between different sets of hadrons.

It is tempting to speculate that the above interpretation of σ may be meaningful for the higher and numerically smaller members of the hierarchy of commutators of T-internal Lie algebras. Should this be the case, one would have an attractive scheme of generating extremely small coupling constants.

APPENDIX A

Internal Symmetries of a Two-Particle System in the Framework of P_0

A system of two free noninteracting particles is represented mathematically by the tensor product state vector

$$|m_1\mathbf{p}_1s_1h_1\rangle \otimes |m_2\mathbf{p}_2s_2h_2\rangle \tag{A-1}$$

This vector is an eigenvector of twelve commuting operators constructed from the basis elements of the Lie algebras $\mathfrak{P}_0(1)$ and $\mathfrak{P}_0(2)$ of the two particles. Explicitly, we diagonalize

$$\begin{aligned} P(i)^2 &= m_i^2 \\ \mathbf{P}(i) &= \mathbf{p}_i \\ W(i)^2 &= -m_i^2 s_i(s_i+1) \\ W_0(i) &= |\mathbf{p}_i| h_i \end{aligned} \tag{A-2}$$

for $i = 1, 2$. The polarization operators $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma$ have the well-known commutation relations

$$\begin{aligned} [P_\mu, W_\nu] &= 0 \\ [W_\mu, W_\nu] &= i\epsilon_{\mu\nu\rho\sigma}P^\rho W^\sigma \\ [M_{\mu\nu}, W_\rho] &= iW_{[\mu}g_{\nu]\rho} \end{aligned}$$

Alternately, a two-particle system may be characterized by the state of its "center of mass" and by the "internal configuration" of the two particles in their c.m. frame. The external or c.m. operators

$$\begin{aligned} P_\mu &= P_\mu(1) + P_\mu(2) \\ M_{\mu\nu} &= M_{\mu\nu}(1) + M_{\mu\nu}(2) \\ W_\mu &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma \end{aligned}$$

obey commutation relations identical to those obeyed by the operators of each of the individual particles. Thus we may simultaneously diagonalize P^2, $\mathbf{P}$, W^2, and W_0; to complete the specification of the state we must construct additional six operators in terms of the basis elements of $\mathfrak{P}_0(1)$ and $\mathfrak{P}_0(2)$. A little experimentation reveals that the operators

$$\begin{aligned} \mu_i &= P(i)^2 \\ \omega_i &= W(i)^2 \\ \lambda_1 &= W(1)\cdot P(2) \\ \lambda_2 &= W(2)\cdot P(1) \end{aligned}$$

commute among themselves and with P_μ, $M_{\mu\nu}$, and of course W_μ. Thus they may simultaneously be diagonalized. The eigenvalues of μ_i and ω_i are given by Eq. (A-2); it remains to investigate those of λ_i.

Consider λ_1. In the c.m. frame $\mathbf{P}(1) = \mathbf{p}_1 = -\mathbf{p}_2 = -\mathbf{P}(2)$. Hence

$$\begin{aligned} \lambda_1 &= W_0(1)P_0(2) - \mathbf{W}(1)\cdot\mathbf{P}(2) \\ &= W_0(1)P_0(2) + \mathbf{W}(1)\cdot\mathbf{P}(1) \\ &= W_0(1)[P_0(1) + P_0(2)] \end{aligned}$$

where we have used $W(1)\cdot P(1) = 0$. Now in the c.m. frame $P_0 = P_0(1) + P_0(2)$ is just m (since $\mathbf{P} = 0$) and $W_0(1) = |\mathbf{p}_1|h_1$. It is readily verified that

$$|\mathbf{p}_1| = \frac{1}{2m}[\Delta(m_1^2, m_2^2, m^2)]^{1/2} \tag{A-3}$$

where

$$\Delta(a,b,c) = a^2 + b^2 + c^2 - 2(ab + bc + ca)$$

so that

$$\lambda_1 = \frac{1}{2}\Delta^{1/2}h_1 \tag{A-4}$$

The three masses and λ_1 are invariant under all transformations of P_0 generated by P_μ and $M_{\mu\nu}$ and hence so is h_1 and, similarly, h_2. Strictly speaking, Eq. (A-4) holds only when applied to state vectors of the form of Eq. (A-1) since, otherwise, neither $\mathbf{P}(1)$ nor $\mathbf{P}(2)$ can be diagonalized (they fail to commute with W_0, e.g.).

Summarizing the preceding discussion, we see that it is possible to introduce the following two-particle state vectors labeled by six external and six internal quantum numbers:

$$|m\mathbf{p}sh; m_1s_1\lambda_1m_2s_2\lambda_2\rangle$$

Under an arbitrary Poincaré transformation (a, l) generated by the external operators P_μ and $M_{\mu\nu}$ only $\mathbf{p}$ and h get mixed, the remaining quantum numbers staying fixed. The natural question arises whether there exist unitary transformations generated by some combinations of $P_\mu(i)$ and $M_{\mu\nu}(i)$ which mix the internal quantum numbers λ_1 and λ_2 (m_1, s_1, m_2, and s_2 are necessarily fixed within the framework of the Poincaré group). Clearly, the generators of these transformations must be Lorentz scalars or invariants of the form

$$A \cdot B = A_\mu B^\mu$$

$$[ABCD] = \varepsilon_{\mu\nu\rho\sigma} A^\mu B^\nu C^\rho D^\sigma$$

and, of course, polynomials of such invariants. Let us denote the set of all hermitian internal operators by $\mathfrak{E}_{\text{int}}$. It is possible to show by direct enumeration that the following fourteen hermitian operators form the basis $\mathfrak{B}$ of $\mathfrak{E}_{\text{int}}$:

$$\begin{array}{lll} P(1)^2 & W(1) \cdot W(2) & W^2 \\ P(2)^2 & W(1) \cdot P(2) & [WPW(1)P(1)] \\ P(1) \cdot P(2) & W(2) \cdot P(1) & [WPW(2)P(2)] \\ W(1)^2 & W(1) \cdot W & [P(1)P(2)W(1)W(2)] \\ W(2)^2 & W(2) \cdot W & \end{array} \tag{A-5}$$

Every element of $\mathfrak{E}_{\text{int}}$ can be written as a linear combination of invariants of the form x, $x \circ y$, $(x \circ y) \circ z$, $\cdots$, with $x, y, z, \cdots \in \mathfrak{B}$; the Jordan product $x \circ y$ is defined by

$$x \circ y = \frac{1}{2}(xy + yx) = (y \circ x) = (x \circ y)^*$$

and reduces to the ordinary operator product whenever x and y commute.

The commutation relations of the operators (Eq. A-5) have the general form

$$[X_i, X_j] = i c_{ij}{}^{kl} X_k \circ X_l$$

for $X_i, X_j, \cdots \in \mathfrak{B}$ with real c's. Clearly, the elements of $\mathfrak{B}$ fail to form a finite-dimensional Lie algebra. Nor does there appear any possibility of generating such algebras by adjoining to $\mathfrak{B}$ polynomials of elements in $\mathfrak{B}$. The last remaining hope is to try to pick out a subset of $\mathfrak{B}$ generating a finite Lie algebra or at least an approximation to it which would resemble any of the approximate particle symmetries observed in nature. This venture too has met with no success. Probably the most serious objection of all to the above method of generating internal symmetries is that the internal quantum numbers we have obtained have a *purely geometrical* interpretation as masses, spins, and helicities. No alternate maximal abelian set of operators appears to be available to replace the one employed above. Thus we must admit defeat and look for other possibilities.

APPENDIX B
Operator Identities

In this appendix we collect some formal operator identities implicitly used in the text. First, we recall the definition of a Lie derivative. For any two operators A, B for which the product AB and BA is defined, the operator

$$\theta(A)B = [A,B] = AB - BA \tag{B-1}$$

is called the Lie derivative of B with respect to A. Higher powers of $\theta(A)$ are defined by induction:

$$\theta^n(A)B = \theta(A)[\theta^{n-1}(A)B], \qquad n \geq 2$$

We also set

$$\theta^0(A)B = B$$

The operator $\theta(A)$ has a number of properties which are simple consequences of its definition (Eq. B-1). We list some of them:

$$\theta(A)A = 0$$

$$\theta(A)B = -\theta(B)A$$

$$\theta(\alpha_1 A_1 + \alpha_2 A_2)B = \alpha_1\theta(A_1)B + \alpha_2\theta(A_2)B \quad (\alpha_1, \alpha_2 \text{ complex})$$

$$\theta(A)(\beta_1 B_1 + \beta_2 B_2) = \beta_1\theta(A)B_1 + \beta_2\theta(A)B_2 \quad (\beta_1, \beta_2 \text{ complex})$$

$$\begin{aligned}\theta(A)(B_1B_2\cdots B_n) &= [\theta(A)B_1]B_2\cdots B_n \\ &\quad + B_1[\theta(A)B_2]\cdots B_n \\ &\quad + \cdots + B_1B_2\cdots[\theta(A)B_n]\end{aligned}$$

$$\theta(A)\theta(B)C = \theta(B)\theta(A)C + \theta(\theta(A)B)C$$

Here, by definition,

$$\theta(A)\theta(B)C = \theta(A)[\theta(B)C]$$

Next, we introduce the exponential operator $E(A)$, depending on the operator A, by setting

$$E(A) = \exp\theta(A) \equiv \sum_{n=0}^{\infty}\frac{1}{n!}\theta^n(A)$$

Some of its properties are the following

$$\begin{aligned}E(A)B &= e^{A}Be^{-A} \\ E(A)E(B)C &= E(A)[E(B)C] \\ E(A)(B_1B_2\cdots B_n) &= [E(A)B_1][E(A)B_2]\cdots[E(A)B_n] \\ E(A)E(B)C &= E(E(A)B)E(A)C \\ E(-A)E(A) &= 1 \\ E^n(A)B &= E(nA)B \qquad n = 0,1,2,\cdots\end{aligned} \tag{B-2}$$

If B is an "eigenvector" of $\theta(A)$ with "eigenvalue" λ, i.e., if

$$\theta(A)B = \lambda B$$

holds, then

$$E(A)B = e^{\lambda}B$$

Similarly, if

$$\theta^2(A)B = \lambda^2 B$$

then

$$E(A)B = \cosh\lambda B + \lambda^{-1}\sinh\lambda\,\theta(A)B$$

If $f(B)$ is an analytic function of the operator B, i.e., if it has an expansion in powers of B,

$$f(B) = \sum_{n=0}^{\infty}\beta_n B^n$$

then

$$E(A)f(B) = f(E(A)B)$$

APPENDIX C
Generalized Hilbert Spaces

Consider the abelian group R of real numbers under addition. The space $L_2(-\infty, \infty)$ of all complex-valued Lebesgue-measurable functions f on $(-\infty, \infty)$ for which

$$\|f\|^2 = \int_{-\infty}^{\infty} dx\, |f(x)|^2 < \infty$$

is a Hilbert space,[33] henceforth denoted by $\mathfrak{H}$. The inner product is given by

$$(f, g) = \int_{-\infty}^{\infty} dx\, f(x)^* g(x) \tag{C-1}$$

for any pair $f, g \in \mathfrak{H}$. If $\alpha \in R$ and $f \in \mathfrak{H}$, then the mapping

$$\begin{aligned} \alpha: \; & f \to T_\alpha f \\ (T_\alpha f)(x) &= f(x+\alpha) \end{aligned} \tag{C-2}$$

is easily seen to be a unitary representation of R on $\mathfrak{H}$. Each $f \in \mathfrak{H}$ thus furnishes a unitary, although in general reducible, representation of R. These representations may be decomposed into irreducible components in a well-known manner (Ref. 58). Namely, one introduces the Fourier transform $\hat{f}$ of a given $f \in \mathfrak{H}$ by

$$\hat{f}(p) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ipx} \tag{C-3}$$

to be understood in the sense of limits in the mean; i.e.,

$$\hat{f}(p) = l.i.m.\, (2\pi)^{-1/2} \int_{-n}^{n} dx\, f(x)\, e^{-ipx} = l.i.m.\, \hat{f}_n(p)$$

if

$$\int dp\, |\hat{f}(p) - \hat{f}_n(p)|^2 \to 0 \qquad \text{as } n \to \infty$$

The function $\hat{f}$ is in $\mathfrak{H}$ and determines f through the inverse transform

$$f(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dx\, \hat{f}(p)\, e^{ipx} \tag{C-4}$$

again in the *l.i.m.* sense. Equation (C-4) provides the desired decomposition of Eq. (C-2) into a continuous direct sum (integral) of one-dimensional unitary representations of R:

$$(T_\alpha f)(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dp\, e^{ip\alpha} \{\hat{f}(p)\, e^{ipx}\} \in \mathfrak{H}$$

The functions $\phi_p(x) = e^{ipx}$ belong to the representation space of R, and, for fixed p, transform irreducibly under R:

$$(T_\alpha \phi_p)(x) = e^{ip\alpha}\, \phi_p(x)$$

However, they are not elements of $\mathfrak{H}$ since their norm $\|\phi_p\|$ is infinite; in physical language, plane wave state vectors are not normalizable. Thus it is unfortunate but true that functions having "nice" transformation properties under R are not in the Hilbert space $\mathfrak{H}$. This circumstance is of course quite general, not at all peculiar to the group R. In fact, the representation space of any noncompact group will contain unnormalizable vectors. As an example, we cite the case of the Lorentz group discussed in Section IV.

In practical applications, it would be very desirable to treat the functions ϕ_p and those in $\mathfrak{H}$ on the same footing. I.e., one would like to extend $\mathfrak{H}$ to a larger space containing the ϕ_p's and equipped with some sort of inner product, much like $\mathfrak{H}$ itself. Indeed, it is possible to achieve this in a rigorous mathematical manner in terms of so-called rigged Hilbert spaces (Ref. 59). Their theory is rather elaborate and requires a number of preliminary mathematical notions which we have no intent to reproduce here. We shall instead formulate the somewhat heuristic concept of a generalized Hilbert space which will amply meet our needs.

Consider the Hilbert space $\mathfrak{H}$ introduced above in connection with representations of group R. Let us adjoin to $\mathfrak{H}$ the eigenvectors ϕ_p of the operator T_α representing elements $\alpha \in R$ and denote the resulting set of functions by $\mathfrak{H}_R$. The inner product in $\mathfrak{H}$ may be extended to functions in $\mathfrak{H}_R$ by relaxing the requirement that (f, g) be a complex-valued function of f and g; now it may be a distribution. We call $\mathfrak{H}_R$ the generalized Hilbert space associated with representations of the group R or simply the R-generalized Hilbert space. Elements $\phi_p \in \mathfrak{H}_R$ are called singular elements of $\mathfrak{H}_R$; their norms are infinite.

[33] Strictly speaking $L_2(-\infty, \infty)$ is a space of classes of functions which differ from each other only on sets of measure zero.

The remaining elements are of finite norm and are called regular elements of $\mathfrak{H}_R$. Every regular element is a (continuous) linear combination of singular elements according to Eq. (C-4), and every singular element is a limit of an almost everywhere convergent sequence of regular elements. An example is furnished by

$$\phi_p(x) = \lim_{n\to\infty} e^{-|x|/n} e^{ipx}$$

APPENDIX D
Rotation and Lorentz Groups

This appendix contains a collection of miscellaneous results from the representation theories of the three-dimensional rotation and the Lorentz groups.

Matrix elements of the unitary operator $U(R)$, Eq. (35), are trivially related to the spherical functions $D^j_{\mu'\mu}$ of the three-dimensional rotation group:

$$\langle j'\mu'|U(R)|j\mu\rangle = \delta_{j'j} D^j_{\mu'\mu}(R)$$

$$D^j_{\mu'\mu}(R) = \langle\mu'|e^{-i\alpha M_3} e^{-i\beta M_2} e^{-i\gamma M_3}|\mu\rangle_j$$

$$= e^{-i\alpha\mu'} d^j_{\mu'\mu}(\beta)\, e^{-i\gamma\mu} \qquad \text{(D-1)}$$

The d-functions are given by Ref. 60:

$$d^j_{\mu'\mu}(\beta) = \sum_\nu (-)^\nu \frac{[(j+\mu')!\,(j-\mu')!\,(j+\mu)!\,(j-\mu)!]^{1/2}}{(j+\mu'-\nu)!\,(j-\mu-\nu)!\,\nu!\,(\nu+\mu-\mu')!}$$

$$\cdot (\cos\beta/2)^{2j+\mu'-\mu-2\nu} (\sin\beta/2)^{2\nu+\mu-\mu'}$$

The Wigner rotation operator $R(l,p)$ for an arbitrary Lorentz transformation $l = (l_\mu{}^\nu)$ and a given four-momentum $p = (p_\mu)$ is defined in Ref. 14:

$$R(l,p) = L(lp)\, l\, L(p)^{-1} \qquad \text{(D-2)}$$

where $L(p)$ is a Lorentz transformation which takes a particle of momentum p to its rest frame:

$$L_\mu{}^\nu(p)\, p_\nu = \tilde{p}_\mu,$$

$$\tilde{p} = (\varepsilon m, \mathbf{0}), \qquad \varepsilon = \operatorname{sgn} p_0,\ m > 0 \qquad \text{(D-3)}$$

Explicit formulas for $L(p)$ and $R(l,p)$ in the spinor representation have been given by Joos (Ref. 61):

$$L(p) \to \varepsilon\,[2m(m + |p_0|)]^{-1/2} (p_0 + \varepsilon m - \mathbf{p}\cdot\boldsymbol{\sigma})$$

$$R(l,p) = (\mu/\mu')^{1/2} [\Re e\,(a_0 + \mathbf{a}\cdot\mathbf{p}/\mu) + i\boldsymbol{\sigma}\cdot\Im m\,(\mathbf{a} + a_0\mathbf{p}/\mu)$$

$$- i\boldsymbol{\sigma}\cdot\mathbf{p}\times(\mathbf{a}\times\mathbf{a}^*)/2\mu]$$

where

$$\mu = \varepsilon(m + |p_0|)$$

$$\mu' = \varepsilon(m + |p'_0|)$$

$$p' = lp$$

and the σ_i being the usual Pauli matrices. The complex quantities a_0 and $\mathbf{a}$ are determined by l through (Ref. 61 and 62)

$$l \to a_0 + \mathbf{a}\cdot\boldsymbol{\sigma} = N\,[\operatorname{tr} l + (l^{0k} - l^{k0} - i\varepsilon^{ijk} l_{ij})\,\sigma_k]$$

$$N = [4 + (\operatorname{tr} l)^2 - \operatorname{tr}(l^2) - i\varepsilon^{\mu\nu\rho\sigma} l_{\mu\nu} l_{\rho\sigma}]^{-1/2}$$

summations on Latin and Greek indices are understood in these formulas. Here $\operatorname{tr} l = l^\mu_\mu$, etc., and

$$\varepsilon^{123} = \varepsilon^{0123} = +1$$

From Eq. (D-2) follow the properties

$$R(l', lp)\, R(l, p = R(l'l, p)$$

$$R(l,p)^{-1} = R(l^{-1}, lp) \qquad \text{(D-5)}$$

We now give a brief discussion of helicity representations of P_0. Let

$$h(\mathbf{p}) = \mathbf{M}\cdot\hat{\mathbf{p}}, \qquad \hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$$

be the helicity operator $(= W_0(\mathbf{P}^2)^{-1/2})$ and introduce its eigenvectors:

$$h(\hat{\mathbf{p}})|psh\rangle = h|psh\rangle$$

If $\mathbf{p}$ has the polar form (p, θ, ϕ), then (Ref. 63 and 64)

$$|psh\rangle = U(H(p))|p_R sh\rangle$$

$$U(H(p)) = e^{-i\phi M_3} e^{-i\theta M_2} e^{i\phi M_3} e^{-i\zeta N_3}$$

$$\zeta = \sinh^{-1}(p/m) \qquad \text{(D-6)}$$

where p_R has an infinitesimal space part in the 3-direction so that $h(\hat{\mathbf{p}}_R) = M_3$. The state vector $U(l)|psh\rangle$ can easily be shown to have momentum lp; hence it must at most be a linear combination of the vectors $|lpsh\rangle$ with different helicities:

$$U(l)|psh\rangle = \sum_{h'} |lpsh'\rangle\langle lpsh'|U(l)|psh\rangle \qquad \text{(D-7)}$$

Using Eq. (D-6), we get

$$\langle lpsh'|U(l)|psh\rangle = \langle p_R sh'|U(H(lp))^{-1}U(l)U(H(p))|p_R sh\rangle \qquad \text{(D-8)}$$

Since the above unitary operator connects two state vectors of a particle at rest, it must represent a pure spatial rotation. We set

$$R_w(l,p) = H(lp)^{-1}\, l\, H(p) \qquad \text{(D-9)}$$

and call R_w the Wigner rotation operator appropriate to helicity representations. Note that R_w is *not* the same as R given by Eq. (D-2). In fact,

$$R_w(l,p) = [L(lp)H(lp)]^{-1} R(l,p)\,[L(p)H(p)]$$

The operators in brackets are spatial rotations as may be seen from the fact that they leave $\tilde{p} = (\epsilon m, \mathbf{0})$ fixed; e.g.,

$$L(p)H(p)p = L(p)p = \tilde{p}$$

The spinor representative of $H(p)$, and hence of $L(p)H(p)$, may be computed from Eq. (D-4). We note that R_w too satisfies the relations of Eq. (D-5).

From Eq. (D-7), (D-8), (D-9), and (D-1) we now find

$$U(l)|psh\rangle = \sum_{h'} |lpsh'\rangle D^{s}_{h'h}(R_w(l,p))$$

whence Eq. (D-8) follows by an application of $U(a)$.

As is well known, every Lorentz transformation l may be factored into a product of two rotations and a pure Lorentz transformation ("boost"; see Ref. 65) along a fixed axis, say the z-axis:

$$l = R'ZR \qquad \text{(D-10)}$$

This factorization is not unique since one has

$$R'ZR = (R'R_3)Z(R_3^{-1}R) \qquad \text{(D-11)}$$

for an arbitrary rotation R_3 about the z-axis. Uniqueness may be secured if we insist that R' always have the form

$$R' = R_3(\alpha')R_2(\beta')$$

where

$$U(R_k(\omega)) = e^{-i\omega M_k} \qquad k = 1,2,3$$

To prove this statement, we use the result (Ref. 66) that every l is uniquely expressible as

$$l = RT$$

where R is a pure rotation and T a boost. It is clear that T may be written as a rotational transform of a boost in the z-direction:

$$T = R_T^{-1}ZR_T \qquad \text{(D-12)}$$

To see the degree of arbitrariness present in this formula, let us suppose that it is valid with R_T replaced by R'_T. Then

$$R_T^{-1}ZR_T = R_T'^{-1}ZR'_T$$

or

$$[Z, R'_T R_T^{-1}] = 0$$

But this means that

$$R'_T R_T^{-1} = R_3$$

or

$$R'_T = R_3 R_T$$

Thus Eq. (D-12) is arbitrary only within a rotation about the z-axis. Now every rotation has a unique factorization of the form

$$R = R_3(\alpha)R_2(\beta)R_3(\gamma), \qquad 0 \leqq \alpha, \gamma < 2\pi, 0 \leqq \beta < \pi$$

in terms of Euler angles. Choosing $R_3(\gamma')R_3 = 1$ in Eq. (D-11) removes the arbitrariness from Eq. (D-10).

Writing out Eq. (D-10) in detail, we have

$$l^{\nu}_{\mu} = [R_3(\alpha')R_2(\beta')Z(\zeta)R_3(\alpha)R_2(\beta)R_3(\gamma)]^{\nu}_{\mu}$$

where

$$R_2(\omega) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & 0 & \sin\omega \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\omega & 0 & \cos\omega \end{bmatrix}$$

$$R_3(\omega) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Z(\zeta) = \begin{bmatrix} \cosh\zeta & 0 & 0 & \sinh\zeta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\zeta & 0 & 0 & \cosh\zeta \end{bmatrix}$$

As discussed in Section IV, the vectors $|k\nu j\mu\rangle$ for certain ranges of values of k, ν, j, and μ form a basis for irreducible unitary representations of the Lorentz group. Spherical functions of this group are defined by the right-hand side of

$$\langle k'\nu' j'\mu' | U(l) | k\nu j\mu\rangle = \delta_{k'k}\,\delta(\nu' - \nu)\,\langle j'\mu' | U(l) | j\mu\rangle_{k\nu}$$

$$l = R'ZR$$

with

$$\langle j'\mu' | U(l) j\mu\rangle_{k\nu} = Q^{k\nu}_{j'\mu';j\mu}(l)$$

$$= \sum_{\mu''} D^{j'}_{\mu'\mu''}(R')\,\mathfrak{Z}^{k\nu\mu''}_{j'j}(\zeta)\,D^{j}_{\mu''\mu}(R)$$

The functions

$$\mathfrak{Z}^{k\nu\mu''}_{j'j}(\zeta) = \langle j' | U(Z) | j\rangle_{k\nu\mu''}$$

have been calculated by Dolginov and Moskalev (Ref. 67) and are given in our notation by

$$\mathfrak{Z}^{k\nu\mu}_{j'j}(\zeta) = \sum_{J=|j-j'|}^{j+j'} (2J+1)\,[(2j'+1)/(2\sigma+1)]^{1/2}$$

$$\cdot\, W(\sigma\sigma + kJj; j'\sigma)$$

$$\times\, C(Jj'j; 0\mu)\, e^{\mu\zeta}\,\Pi_J(\nu, \zeta)$$

Here

$$2\sigma + 1 = i\nu$$

$$\Pi_J(\nu, \zeta) = \frac{\sinh^J \zeta}{M_J}\,\frac{d^{J+1}\cos\nu\zeta}{d(\cosh\zeta)^{J+1}}$$

$$M_J = \prod_{n=0}^{J} (\nu^2 + n^2)^{1/2}$$

The C's are the usual Clebsch-Gordan coefficients, and the W's are Racah functions (Ref. 68).

APPENDIX E

Completeness of $\mathfrak{P}$

We recall the definition of a complete Lie algebra given in Section IV:

A Lie algebra $\mathfrak{L}$ is said to be complete if and only if each of its automorphisms continuously connected to the identity automorphism is generated by some element of the enveloping algebra $\mathfrak{E}$ of $\mathfrak{L}$.

To show that $\mathfrak{P}$ is complete, we examine its automorphisms, one by one. The most general linear transformation of $M_{\mu\nu}$ has the form

$$M_{\mu\nu} \rightarrow M'_{\mu\nu} = a_{\mu\nu}{}^{\rho\sigma} M_{\rho\sigma} + b_{\mu\nu}{}^{\rho} P_\rho + c_{\mu\nu}{}^{\rho} X_\rho + d_{\mu\nu} I \qquad \text{(E-1)}$$

The quantities a, b, c, d are antisymmetric in μ and ν and are assumed to be continuous functions of a parameter, say t, with the properties

$$a_{\mu\nu}{}^{\rho\sigma} = \delta^{\rho}_{\mu}\,\delta^{\sigma}_{\nu}$$

$$b_{\mu\nu}{}^{\rho} = c_{\mu\nu}{}^{\rho} = d_{\mu\nu} = 0$$

for $t = 0$. In the most general case, b, c, and d must be independent of each other. The first three terms in Eq. (E-1) are disposed of immediately by noting that they are generated by applying $\exp[i\theta(A)]$ to $M_{\mu\nu}$ with A in turn proportional to $M_{\alpha\beta}$, P_α, and X_α. There is no continuous automorphism yielding the last term in

Eq. (E-1). For suppose there were one. Then, schematically, we should have

$$[M', M'] = iM'$$

But $M' = aM + dI$ gives

$$[M', M'] = ia^2 M \neq iM'$$

a contradiction.

Next, consider the transformations[34]

$$P_\mu \rightarrow P'_\mu = a_\mu{}^\nu P_\nu + b_\mu{}^\nu X_\nu + c_\mu I + d_\mu{}^{\nu\rho} M_{\nu\rho} \tag{E-2}$$

The first three terms are obtained by applying to P_μ the operator $\exp[i\theta(A)]$ with A in turn proportional to $M_{\alpha\beta}$, $X_\alpha X_\beta / I$, and X_α. We show now that there is no automorphism of $\mathfrak{P}$ yielding the last term in Eq. (E-2). We set $b = c = 0$ in Eq. (E-2) and rewrite this transformation in the more convenient form

$$P_\mu \rightarrow P'_\mu = e^{i\theta(At)} P_\mu = a_{\mu\nu}(t) P^\nu + b_{\mu\rho\nu}(t) M^{\nu\rho} \tag{E-3}$$

for some A in the enveloping algebra of $\mathfrak{P}$. Here

$$a_{\mu\nu}(t) = g_{\mu\nu} + \alpha_{\mu\nu} t + 0(t^2)$$
$$b_{\mu\nu\rho}(t) = \beta_{\mu\nu\rho} t + 0(t^2)$$

and

$$b_{\mu\nu\rho}(t) = -b_{\mu\rho\nu}(t)$$
$$\beta_{\mu\nu\rho} = -\beta_{\mu\rho\nu} \tag{E-4}$$

To order t, we have

$$P'_\mu = P_\mu + (\alpha_{\mu\nu} P^\nu + \beta_{\mu\nu\rho} M^{\nu\rho}) t + 0(t^2)$$
$$= P_\mu + i[A, P_\mu] t + 0(t^2)$$

or

$$[A, P_\mu] = -i(\alpha_{\mu\nu} P^\nu + \beta_{\mu\nu\rho} M^{\nu\rho})$$

Using the Jacobi identity for the triple A, P_μ, P_ν, we find

$$(\beta_{\mu\rho\nu} - \beta_{\nu\rho\mu}) P^\rho = 0$$

or

$$\beta_{\mu\nu\rho} = \beta_{\rho\nu\mu} \tag{E-5}$$

Using Eq. (E-4) and (E-5) repeatedly we get

$$\beta_{\mu\nu\rho} = \beta_{\rho\nu\mu} = -\beta_{\rho\mu\nu} = -\beta_{\nu\mu\rho}$$
$$= \beta_{\nu\rho\mu} = \beta_{\mu\rho\nu} = -\beta_{\mu\nu\rho}$$

Thus $\beta_{\mu\nu\rho} \equiv 0$ and hence Eq. (E-3) cannot be a continuous automorphism of $\mathfrak{P}$ yielding terms proportional to $M_{\mu\nu}$.

The transformations $X_\mu \rightarrow X'_\mu$ are reduced to those of P_μ by the duality between P and X. This completes the proof of completeness of $\mathfrak{P}$.

[34]We ignore the trivially generated scale transformations (Sec. IV).

APPENDIX F

Transformation Coefficients Between the Basis Vectors of $\mathfrak{H}(P)$ and $\mathfrak{H}(P'')$

In this appendix we shall compute the transformation coefficients between the basis vectors $|\sigma k \nu p s h\rangle$ and $|\sigma k \nu p j \mu\rangle$ spanning representation Hilbert spaces of groups P and P''. Both vectors are eigenvectors of the same set of commuting operators save two, namely W^2 and $W_0(\mathbf{P}^2)^{-1/2}$ vs $\mathbf{S}^2$ and S_3. It is therefore clear that the transformation coefficients will have the form

$$\langle \sigma k \nu p j \mu | \sigma' k' \nu' p' s h\rangle = \delta(\sigma - \sigma')\,\delta_{kk'}\,\delta(\nu - \nu')\,\delta(p - p')\,\mathrm{M}^{k\nu\sigma}_{j\mu;sh}(p) \tag{F-1}$$

Consider now the quantity

$$X = \langle p' j \mu | \exp(-i\omega{:}M/2) | psh\rangle_\lambda$$

where λ collectively denotes the quantum numbers $\{\sigma, k, \nu\}$. Assuming that $p^2, p_0 > 0$, we have

$$X = \sum_{h'} D^s_{h'h}(R_w(l,p))\,\langle p' j \mu | lpsh'\rangle_\lambda \tag{F-2}$$

where $l = e^\omega$. On the other hand,

$$\exp(-i\omega{:}M/2) = \exp(-i\omega{:}S/2)\exp(-i\omega{:}L/2)$$

according to Eq. (22). Thus

$$\begin{aligned} X &= (\exp(i\omega{:}S/2)\,\phi(p' j \mu), \exp(-i\omega{:}L/2)\,\phi(psh))_\lambda \\ &= \sum_{j'\mu'} Q^{k\nu}_{j\mu;j'\mu'}(l)\,\langle p' j' \mu' | \exp(-i\omega{:}L/2) | psh\rangle_\lambda \end{aligned}$$

The action of $\exp(-i\omega{:}L/2)$ on $|psh\rangle_\lambda$ is quite complicated. However, it is quite simple to see its effect on $|p' j' \mu'\rangle_\lambda$ since the operator L commutes with all the operators in which this vector is diagonal except P_μ. Now

$$P_\mu \exp(i\omega{:}L/2)|p' j' \mu'\rangle_\lambda = (l^{-1}p')_\mu \exp(i\omega{:}L/2)|p' j' \mu'\rangle_\lambda$$

Thus we may set

$$\exp(i\omega{:}L/2)|p' j' \mu'\rangle_\lambda = |l^{-1}p' j' \mu'\rangle_\lambda$$

It follows that

$$X = \sum_{j'\mu'} Q^{k\nu}_{j\mu;j'\mu'}(l)\,\langle l^{-1}p' j' \mu' | psh\rangle_\lambda \tag{F-3}$$

Comparing Eq. (F-2) and (F-3), multiplying through by $D^s_{hh''}(R^{-1}_w(l,p))$ summing on h, and then dropping the primes on h'', we find

$$\langle p' j \mu | lpsh\rangle_\lambda = \sum_{j'\mu'h'} Q^{k\nu}_{j\mu;j'\mu'}(l) D^s_{h'h}(R^{-1}_w(l,p))\,\langle l^{-1}p' j' \mu' | psh'\rangle_\lambda$$

This result shows how transformation coefficients behave under Lorentz transformations. If we knew a coefficient in a *particular* frame, then the above formula would give it for arbitrary vectors p which can be reached from this particular frame by proper orthochronous Lorentz transformations. For this purpose, consider the special case of $p = p_R = (m, \epsilon \mathbf{e}_3)$, $\epsilon \to 0^+$. One finds

$$W^2 = m^2\mathbf{S}^2$$

$$W_0(\mathbf{P}^2)^{-1/2} = S_3$$

Thus

$$\langle l^{-1}p' j' \mu' | p_R s h'\rangle_\lambda = \delta(l^{-1}p' - p_R)\,\delta_{j's}\,\delta_{\mu'h'}$$

and comparing with Eq. (F-1), we find

$$M^{k\nu}_{j\mu;sh}(p) = \sum_{h'} Q^{k\nu}_{j\mu;sh'}(l)\,D^s_{h'h}(R^{-1}_w(l, l^{-1}p))$$

where we have now set $lp_R = p$. We see that the M-function is independent of σ; accordingly, we have omitted this label. The Lorentz transformation l is determined wholly by p: l^{-1} takes p to its rest frame with $\mathbf{p}_R = \epsilon \mathbf{e}_3$. Similar expressions for the M-functions may be derived for spacelike and lightlike momenta p; however, we shall have no occasion to use them and hence omit their derivation.

We next wish to consider the transformation properties of the states $|\sigma k\nu psh\rangle$ under the unitary transformation $U(v) = \exp(-iv \cdot X)$. We have

$$U(v)\,|psh\rangle_\lambda =$$

$$\int dp' \int dp'' \int dp''' \sum_{\substack{s'j'j \\ h'\mu'\mu}} |p's'h'\rangle_\lambda \langle p's'h'|p''j'\mu'\rangle_\lambda$$

$$\cdot \langle p''j'\mu'|\,U(v)\,|p'''j\mu\rangle_\lambda \langle p'''j\mu|psh\rangle_\lambda$$

$$= \int dp' \sum_{\substack{s'j'j \\ h'\mu'\mu}} |p's'h'\rangle_\lambda M^{k\nu}_{j'\mu';s'h'}(p')^*$$

$$\cdot \langle p'j'\mu'|\,U(v)\,|pj\mu\rangle_\lambda M^{k\nu}_{j\mu;sh}(p)$$

But by Eq. (31)

$$U(v)\,|pj\mu\rangle_\lambda = |p + \sigma v\; j\mu\rangle_\lambda$$

Thus

$$\langle p'j'\mu'|\,U(v)\,|pj\mu\rangle_\lambda = \delta(p' - p - \sigma v)\,\delta_{j'j}\,\delta_{\mu'\mu}$$

and so

$$U(v)\,|psh\rangle_\lambda$$

$$= \sum_{\substack{s'j \\ h'\mu}} |p + \sigma v s'h'\rangle_\lambda M^{k\nu}_{j\mu;s'h'}(p + \sigma v)^* M^{k\nu}_{j\mu;sh}(p)$$

Now

$$\sum_{j\mu} M^{k\nu}_{j\mu;s'h'}(p + \sigma v)^* M^{k\nu}_{j\mu;sh}(p)$$

$$= \sum_{h_1 h_2} \sum_{j\mu} Q^{k\nu}_{j\mu;s'h_2}(l_2)^* D^{s'}_{h_2 h'}(R_2)^* Q^{k\nu}_{j\mu;sh_1}(l_1)\, D^{s}_{h_1 h}(R_1)$$

$$= \sum_{h_1 h_2} D^{s'}_{h' h_2}(R^{-1}_2)\, Q^{k\nu}_{s'h_2;sh_1}(l^{-1}_2 l_1)\, D^{s}_{h_1 h}(R_1)$$

$$\equiv \mathfrak{M}^{k\nu\sigma}_{s'h';sh}(p, v)$$

where

$$R_1 = R^{-1}_w(l_1,\, l^{-1}_1 p)$$

$$R_2 = R^{-1}_w(l_2,\, l^{-1}_2(p + \sigma v))$$

$$l^{-1}_1 p = (m, \varepsilon \mathbf{e}_3)$$

$$l^{-1}_2(p + \sigma v) = (m', \varepsilon \mathbf{e}_3)$$

$$m'^2 = (p + \sigma v)^2$$

Thus

$$U(v)\,|psh\rangle_\lambda = \sum_{s'h'} \mathfrak{M}^{k\nu\sigma}_{s'h';sh}(p, v)\,|p + \sigma v s'h'\rangle_\lambda$$

This formula is valid provided the four-vector $p + \sigma v$ is of the same kind as p, i.e., $(p + \sigma v)^2 > 0$ and $(p + \sigma v)_0 > 0$. If the transformation $U(v)$ takes p into a different kind of vector, then one must modify the $\mathfrak{M}$-functions by replacing the D's by spherical functions appropriate to the little group of the transformed vector $p + \sigma v$. The same procedure must be used if the initial vector p is not of the type considered above.

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